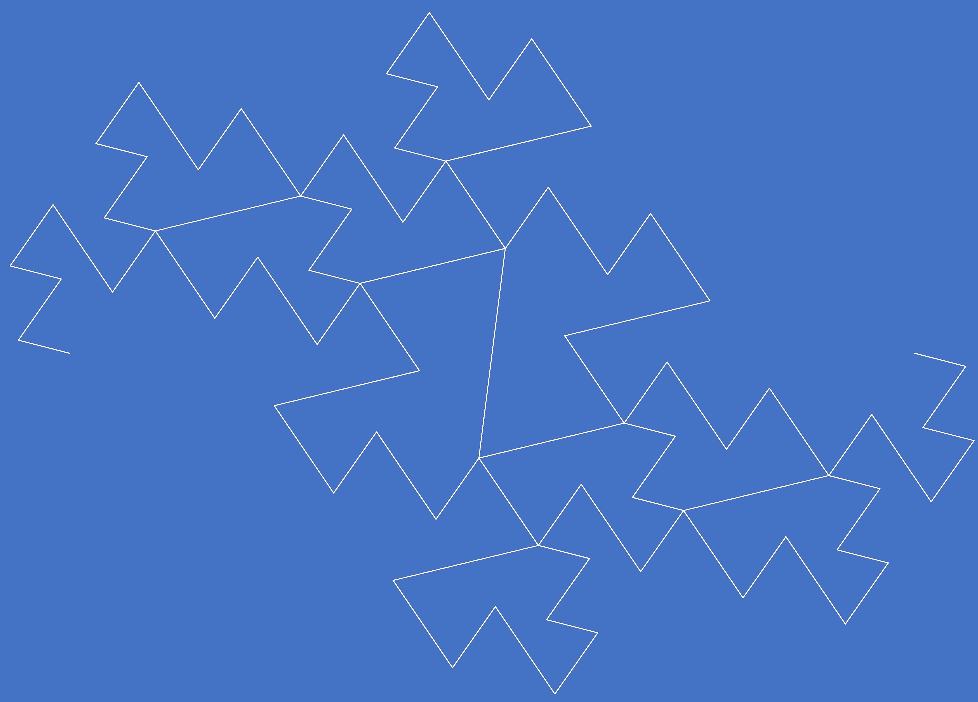
PLANE-FILLING FRACTALS WITH OTHER GRIDS



Nico Bakker

Table of contents

Introduction

Much has been published about plane-filling fractals. Well-known plane-filling fractals are the Terdragon and the 5-Dragon. These two, as well as many other plane-filling fractals, are based on a triangular or square grid.

I asked myself the following questions:

1. Are there also plane-filling fractals that are not tied to a triangular or square grid?
2. If so, what are the properties of such fractals?

The result of this search can be found in this study.

Before I start with this, I will show the aforementioned Terdragon and the 5-Dragon and their dimension.

Then I show the fractals I found, and how I got there. I have taken the liberty of naming these new fractals.

Nico Bakker
Hoorn, august 2025

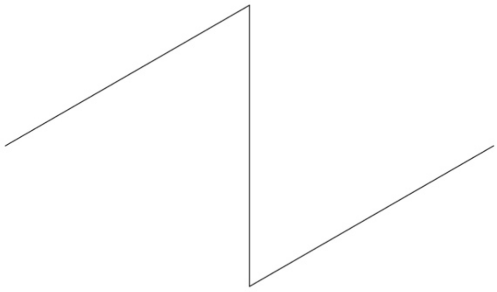
Known plane-filling fractals

Of the well-known plane-filling fractals, I have chosen the Terdragon and the 5-Dragon to serve as an introduction to fractals and to illustrate calculations to the dimension.

Terdragon

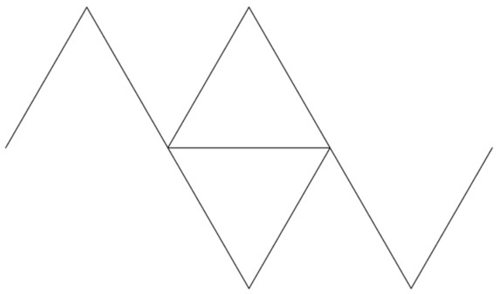
In the construction of a fractal, the line segment with length 1 (step 0) is replaced by a number of contiguous line segments. In the case of the Terdragon, these are three line segments with length $\frac{1}{3}\sqrt{3}$, such that the middle line segment is vertical.

step 0

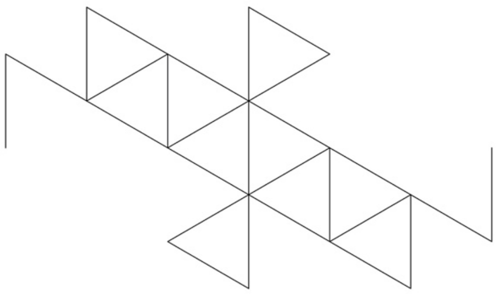


step 1

In step 2, each of the three line segments of step 1 is replaced by three line segments in the same way. In the same way, step 3 was created after processing all 9 line segments of step 2.

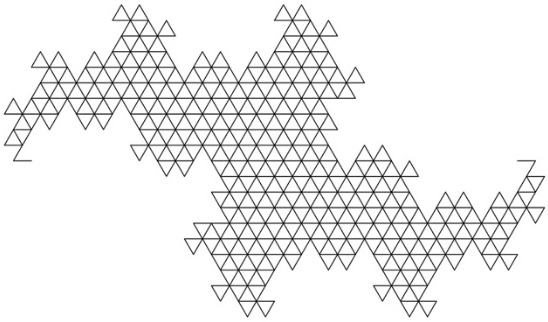


step 2

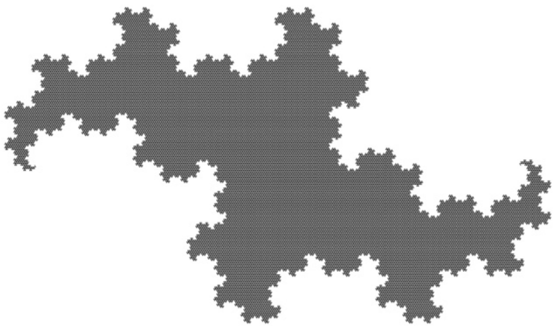


step 3

Here are steps 6 and 10:



step 6



step 10

The line segments become smaller with each subsequent step. The reduction factor for the Terdragon is $\sqrt{3}$. The number of line segments triples with each subsequent step.
Because with each subsequent step all line segments are processed in the same way as with the transition from step 0 to step 1, step 1 determines what the fractal will eventually look like.

A fractal is plane-filling if two conditions are met:
1. the dimension is 2, and
2. The line segments do not intersect or overlap.

Dimension

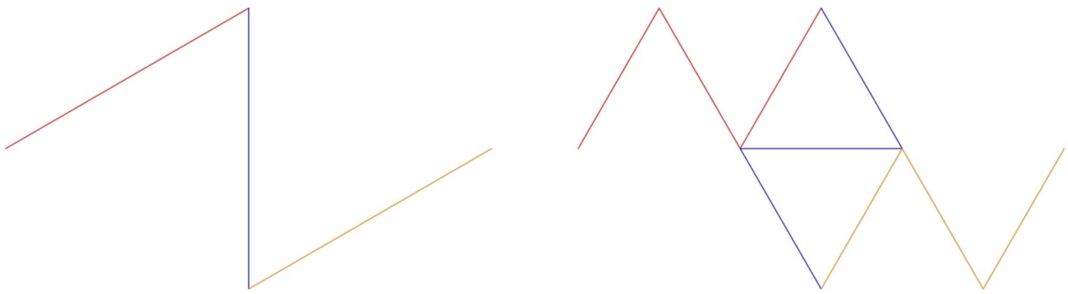
The dimension of a fractal of which the line segments in step 1 all have the same length can be calculated using the Haussdorf formula:

$$d = \frac{\log(\text{aantal kopiëen})}{\log(\text{verkleiningsfactor})}$$

For the Terdragon, that will be: $d = \frac{\log(3)}{\log(\sqrt{3})} = 2$. Condition 1 is therefore met.

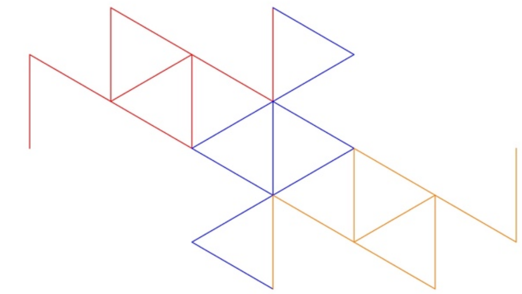
Cutting and overlapping

To show that the line segments of a fractal do not intersect or overlap, the line segments of step 1 are provided with different colors:

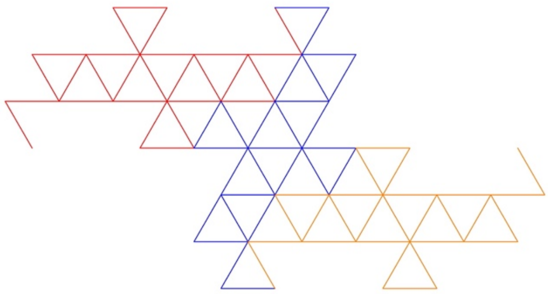


step 1 step 2

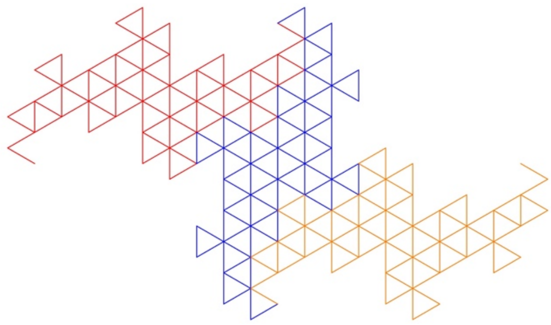
Then in step 2 the colors are maintained. Three times step 1 is then easily recognizable. It can also be seen that the fractal only touches itself in the vertices. There are no intersecting or overlapping line segments.
In step 3 a step 2 is recognizable three times, in step 4 three times a step 3, and so on.



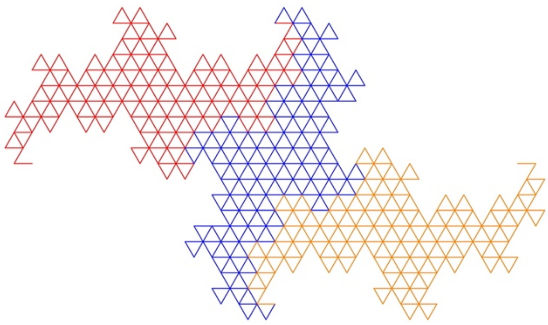
step 3



step 4



step 5



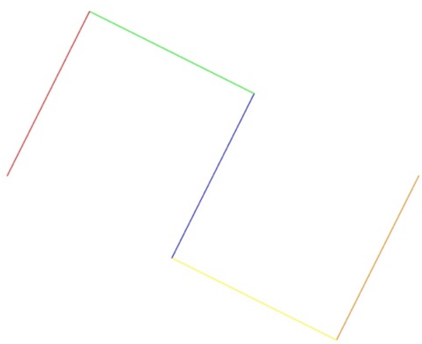
step 6

It is clear that each step consists of three copies of the previous step, and that the left and right copies touch the middle copy only at the vertices. This means that there are no intersecting or overlapping line segments.

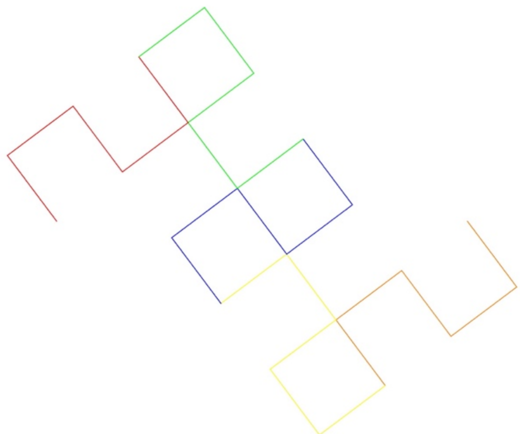
Both conditions are met and so the Terdragon fractal is plane-filling.

5-Dragon

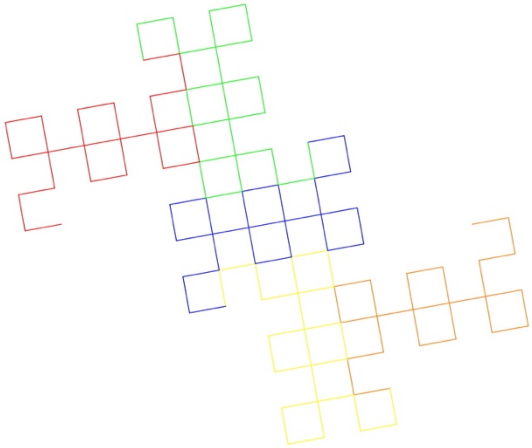
In the 5-Dragon, the line segment with length 1 (step 0) is replaced by five line segments with length $\frac{1}{5}\sqrt{5}$, in such a way that the line segments are perpendicular to each other. The line segments are immediately shown in color; I leave out step 0.



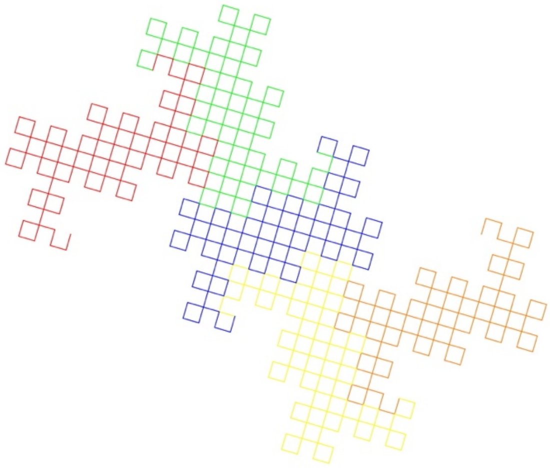
step 1



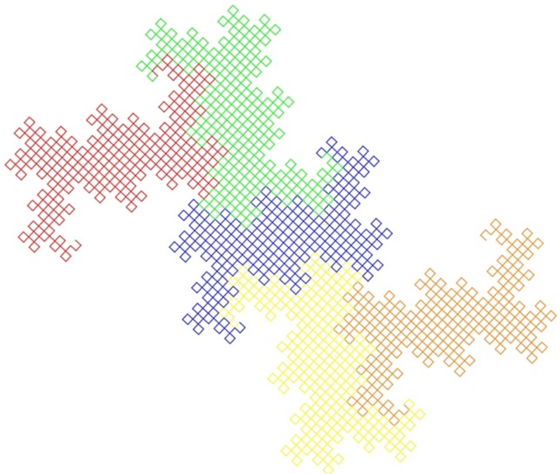
step 2



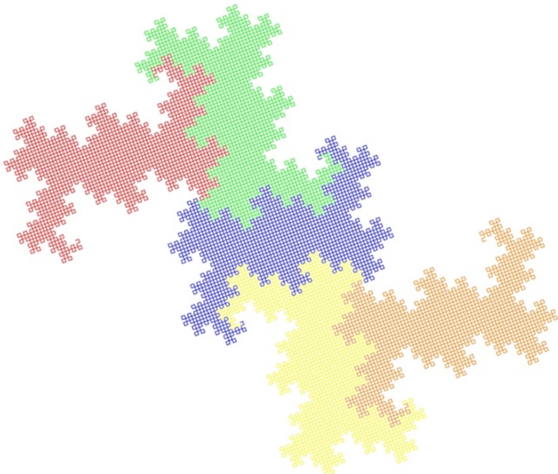
step 3



step 4



step 5



step 6

The reduction factor for the 5-Dragon is $\sqrt{5}$. The number of line segments increases fivefold with each subsequent step.

Dimension

The dimension of a fractal of which the line segments in step 1 all have the same length can be calculated using the Haussdorf formula.

For the 5-Dragon, that will be: $d = \frac{\log(5)}{\log(\sqrt{5})} = 2$. The first condition is therefore satisfied.

Cutting and overlapping

The pictures in color above show that in each step the five copies of the previous step only touch each other at the vertices. This means that there are no intersecting or overlapping line segments.

Both conditions are met and so the 5-Dragon fractal is plane-filling.

Duck curve

To find fractals that are not in a square or triangular grid, there are a number of considerations:

1. The fractal must have symmetry (otherwise it will soon become chaos)
2. The fractal must have a dimension 2
3. The line segments do not all have to be the same length

These last two points require a different method to calculate the dimension of a fractal, because Haussdorf's formula only applies to fractals whose line segments are all the same length.

Dimension calculation

Mandelbrot does have a suggestion for this calculation in his book "The Fractal Geometry of Nature" (pp. 56 and 57). He reasons as follows:

a. if the dimension is 1 (i.e. the fractal lies on a straight line), then the sum of the line segments is 1 (i.e. the distance between $(0,0)$ and $(1,0)$). In formula: $r_1 + r_2 + \ldots + r_n = 1$, or $\sum_{m=1}^{n} r_m = 1$, where r_m are the lengths of the line segments.

b. in a fractal with n equal line segments with length r is the reduction factor $\frac{1}{r}$.
The dimension is then $d = \frac{\log(n)}{\log(\frac{1}{r})} = {}^{\frac{1}{r}}log(n)$.
This can be rewritten as: $\left(\frac{1}{r}\right)^d = n \quad \rightarrow \quad r^d = \frac{1}{n} \quad \rightarrow \quad nr^d = 1$, or $\sum_{m=1}^{n} r^d = 1$.

By combining the two above arguments, Mandelbrot comes to the suggestion, that the dimension of a fractal with line segments of unequal length can be calculated with the dimension-generating function $G(d) = \sum_{m=1}^{n} {r_m}^d$, where the dimension d is the unique real root of the equation $G(d) = 1$.

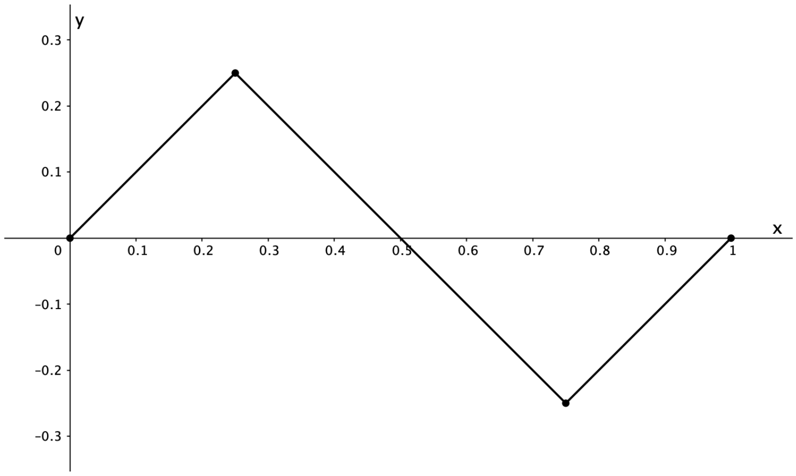
In other words: calculate the dimension d from the equation $\sum_{m=1}^{n} {r_m}^d = 1$.

Mandelbrot has no evidence for this, but in all cases where he has applied it, it is correct.

Example

Mandelbrot gives an example (page 67), where the first step of the fractal looks like the following:

The vertices are on $(0, 0)$, $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{3}{4}, -\frac{1}{4}\right)$ and $(1, 0)$. The lengths of the line segments are respectively. $\frac{1}{4}\sqrt{2}$, $\frac{1}{2}\sqrt{2}$ and $\frac{1}{4}\sqrt{2}$.



To get the dimension d we solve the equation

$\left(\frac{1}{4}\sqrt{2}\right)^d + \left(\frac{1}{2}\sqrt{2}\right)^d + \left(\frac{1}{4}\sqrt{2}\right)^d = 1.$

Remember, that $\frac{1}{2}\sqrt{2} = 2^{-1} \cdot 2^{\frac{1}{2}} = 2^{-\frac{1}{2}}$ and $\frac{1}{4}\sqrt{2} = 2^{-2} \cdot 2^{\frac{1}{2}} = 2^{-1\frac{1}{2}} = \left(2^{-\frac{1}{2}}\right)^3$.

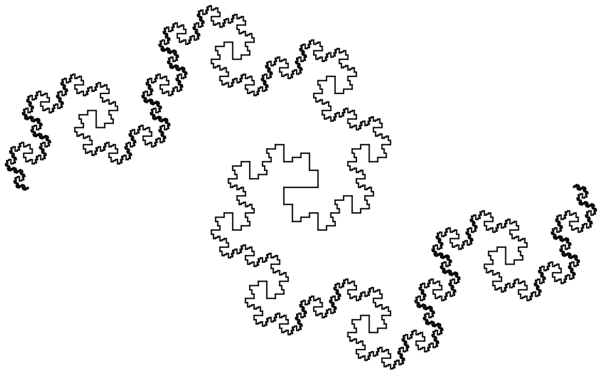
$$\left(2^{-\frac{1}{2}}\right)^{3d} + \left(2^{-\frac{1}{2}}\right)^{d} + \left(2^{-\frac{1}{2}}\right)^{3d} = 1 \quad \rightarrow \quad 2^{-\frac{d}{2}} + 2 \cdot 2^{-\frac{3d}{2}} = 1$$

$$2^{\frac{2d}{2}} + 2 = 2^{\frac{3d}{2}} \quad \rightarrow \quad \left(2^{\frac{d}{2}}\right)^3 - \left(2^{\frac{d}{2}}\right)^2 - 2 = 0$$

This is a cubic equation of $2^{\frac{d}{2}}$. Using Cardano's (or internet's) formula, this provides:

$2^{\frac{d}{2}} \approx 1{,}69562077 \quad \rightarrow \quad d \approx 2 \cdot {}^{2}log 1{,}69562077 \approx 1{,}5236.$

On the right we see step 8 of this fractal. The dimension of about 1.5 may well be correct, because this fractal is certainly not plane-filling (dimension 2) and also far from a straight line (dimension 1).



Plane-filling fractals

Calculating the dimension can lead to solving tricky equations. Now, in our search for plane-filling fractals, we want the fractal to have a dimension 2.

We fill in $d = 2$ and so get $\sum_{m=1}^{n} {r_m}^2 = 1$, as an additional condition for calculating the lengths of the line segments.

Plane-filling fractal with three line segments

To start, let's move on to the example above of Mandelbrot. We take a fractal that starts in the first step and ends with two line segments of equal length and in between a line segment of which the middle is on $\left(\frac{1}{2}, 0\right)$. We do this to give the fractal symmetry.

We call the length of the middle line segment *a* and of the other two *b*. For these lengths, the following applies: $b^2 + a^2 + b^2 = 1$, ofwel $a^2 + 2b^2 = 1$. This means that if we take a value for *a* , we can calculate the value of *b* , and vice versa.
We prefer to work with a ratio between *a* and *b* and that is why we propose: $a = k \cdot b$, where the factor *k* indicates how many times greater *a* is than *b*. We then get:

$$(kb)^2 + 2b^2 = 1 \quad \rightarrow \quad k^2b^2 + 2b^2 = 1 \quad \rightarrow \quad b^2(k^2+2) = 1$$

$$b^2 = \frac{1}{k^2+2} \quad \rightarrow \quad b = \sqrt{\frac{1}{k^2+2}} = \frac{1}{\sqrt{k^2+2}}$$

$$a = kb = \frac{k}{\sqrt{k^2+2}}$$

The cosine rule in the upper triangle gives:

$$(\tfrac{1}{2}a)^2 = (\tfrac{1}{2})^2 + b^2 - 2 \cdot \tfrac{1}{2} \cdot b \cdot \cos\alpha$$

$$\frac{k^2}{4(k^2+2)} = \frac{1}{4} + \frac{1}{k^2+2} - \frac{1}{\sqrt{k^2+2}}\cos\alpha$$

$$\frac{1}{\sqrt{k^2+2}}\cos\alpha = \frac{1}{4} + \frac{1}{k^2+2} - \frac{k^2}{4(k^2+2)}$$

$$\cos\alpha = \left[\frac{k^2+2}{4(k^2+2)} + \frac{4}{4(k^2+2)} - \frac{k^2}{4(k^2+2)}\right]\sqrt{k^2+2} = \frac{6}{4(k^2+2)}\sqrt{k^2+2} = \frac{3}{2\sqrt{k^2+2}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{2\sqrt{k^2+2}}\right)^2} = \sqrt{1-\frac{9}{4(k^2+2)}}$$

$$= \sqrt{\frac{4(k^2+2)}{4(k^2+2)} - \frac{9}{4(k^2+2)}} = \sqrt{\frac{4k^2+8-9}{4(k^2+2)}} = \frac{1}{2}\sqrt{\frac{4k^2-1}{k^2+2}}$$

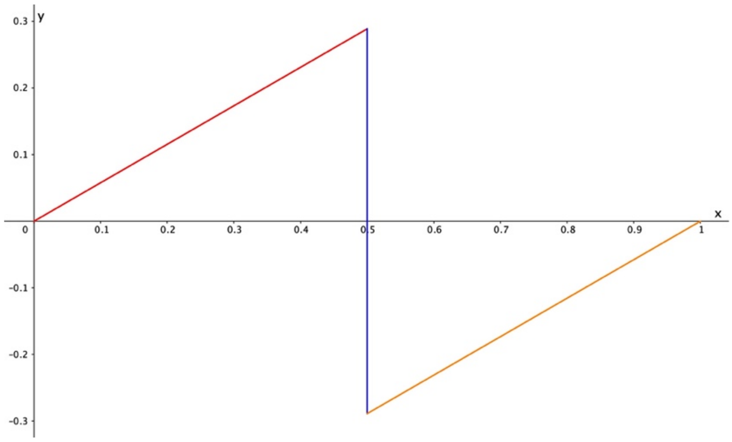
$$x = b \cdot \cos\alpha = \frac{1}{\sqrt{k^2+2}} \cdot \frac{3}{2\sqrt{k^2+2}} = \frac{3}{2(k^2+2)}$$

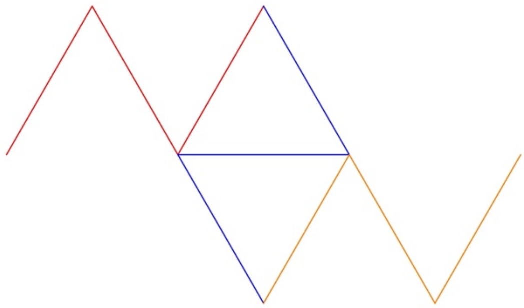
$$y = b \cdot \sin\alpha = \frac{1}{\sqrt{k^2+2}} \cdot \frac{1}{2}\sqrt{\frac{4k^2-1}{k^2+2}} = \frac{\sqrt{4k^2-1}}{2(k^2+2)}$$

We now have formulas for *a*, *b*, *x* and *y*, expressed in *k*. We are going to experiment a bit by filling in a number of values for *k*.

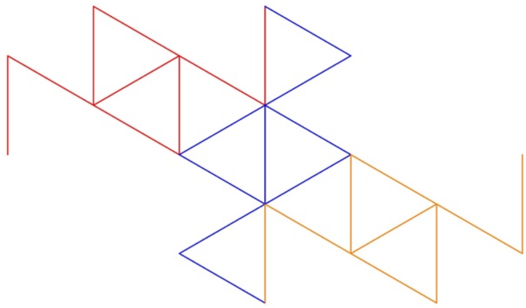
$k = 1$

With $k = 1$ we get an already known fractal, namely the Terdragon, of which we have already seen that it is plane-filling.





step 2



step 3

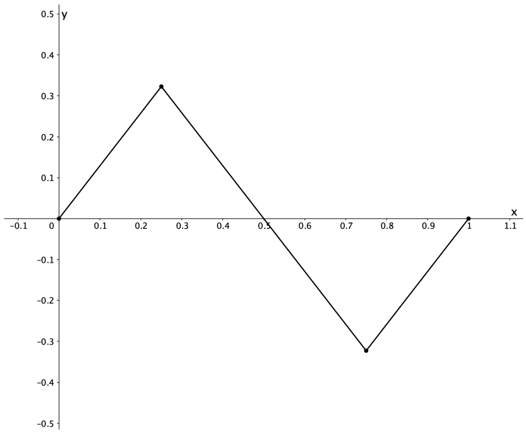
$k = 2$

With $k = 2$ we have the same proportions in the lengths of the line segments as in Mandelbrot's example. The difference is that we now have a dimension 2.

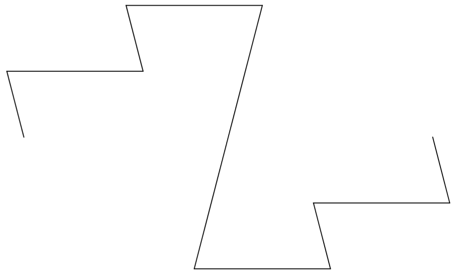
$a = \frac{k}{\sqrt{k^2+2}} = \frac{2}{\sqrt{2^2+2}} = \frac{1}{3}\sqrt{6}$ and $b = \frac{1}{\sqrt{k^2+2}} = \frac{1}{\sqrt{2^2+2}} = \frac{1}{6}\sqrt{6}$.

The coordinates of the vertices are: $(0, 0)$, $\left(\frac{1}{4}, \frac{1}{12}\sqrt{15}\right)$, $\left(\frac{3}{4}, -\frac{1}{12}\sqrt{15}\right)$ and $(1, 0)$.

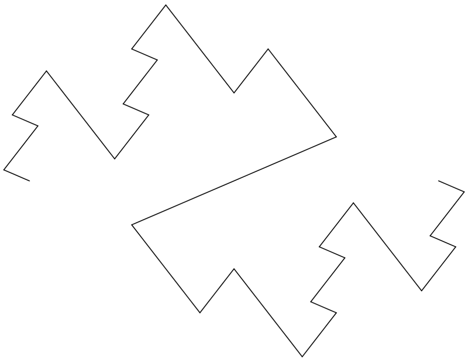
An interesting phenomenon presents itself at step 4: the fractal touches itself in two vertices. At steps 5 and 6, the fractal touches itself in even more vertices. Unfortunately, at step 6 we also see that line segments start to intersect, and that becomes even more numerous in subsequent steps. This fractal is therefore not plane-filling.



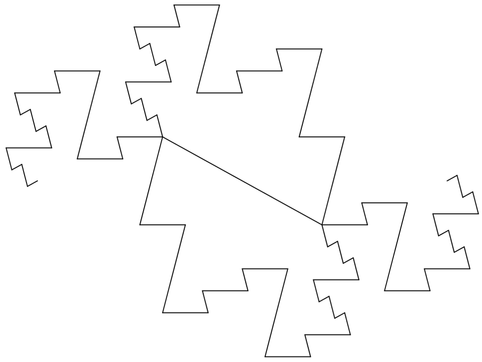
step 1



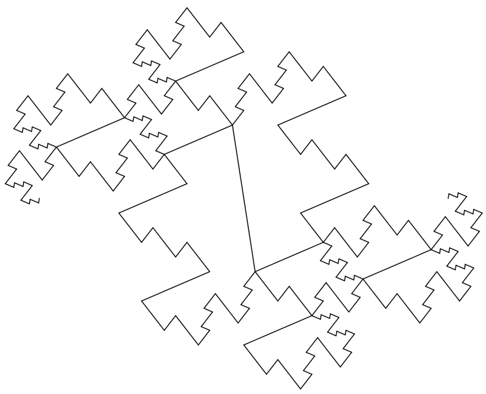
step 2



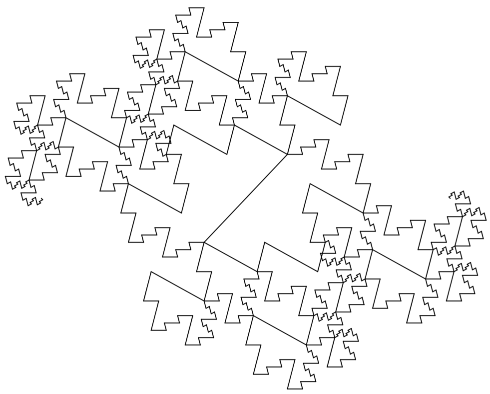
step 3



step 4



step 5



step 6

$k = \sqrt{2}$

With $k = \sqrt{2}$ we get:

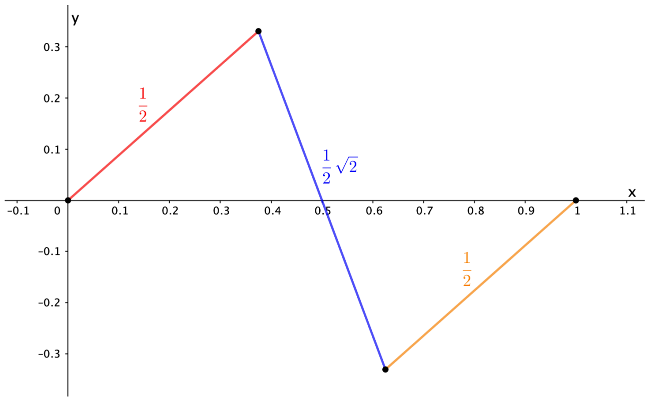
$$a = \frac{k}{\sqrt{k^2+2}} = \frac{\sqrt{2}}{\sqrt{2+2}} = \frac{1}{2}\sqrt{2} \text{ and}$$

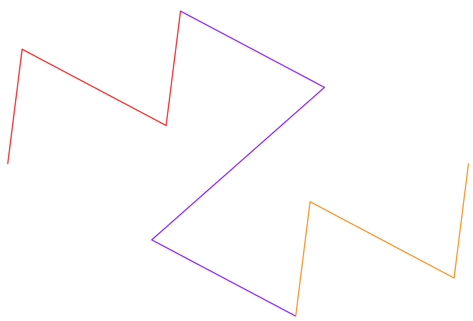
$$b = \frac{1}{\sqrt{k^2+2}} = \frac{1}{\sqrt{2+2}} = \frac{1}{2}.$$

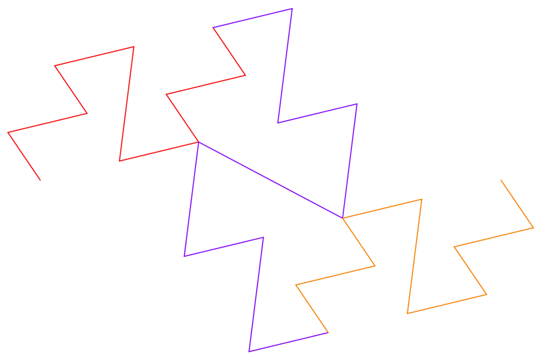
The coordinates of the vertices are:

$(0,0)$, $\left(\frac{3}{8}, \frac{1}{8}\sqrt{7}\right)$, $\left(\frac{5}{8}, -\frac{1}{8}\sqrt{7}\right)$ and $(1,0)$.

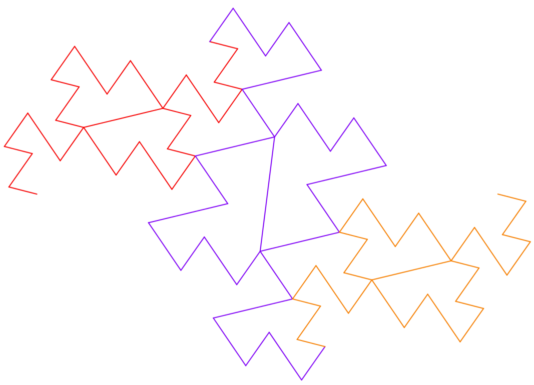
Step 1 looks like the one on the right.

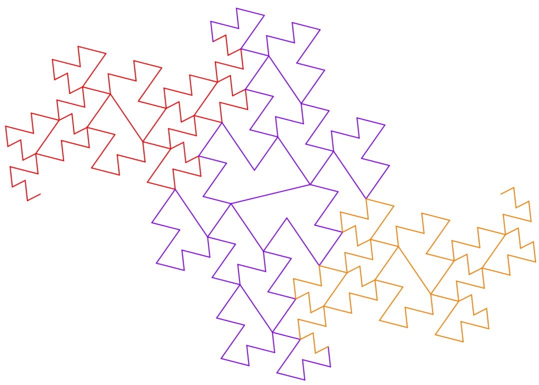


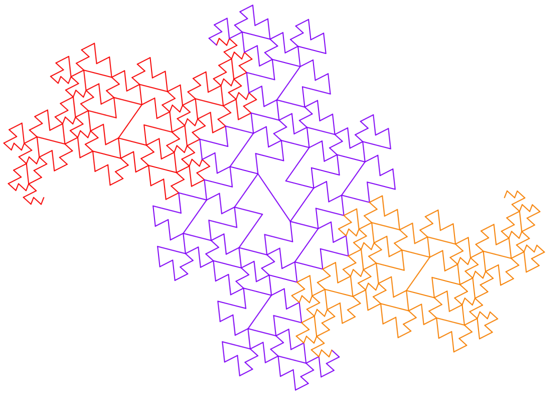

step 2


step 3

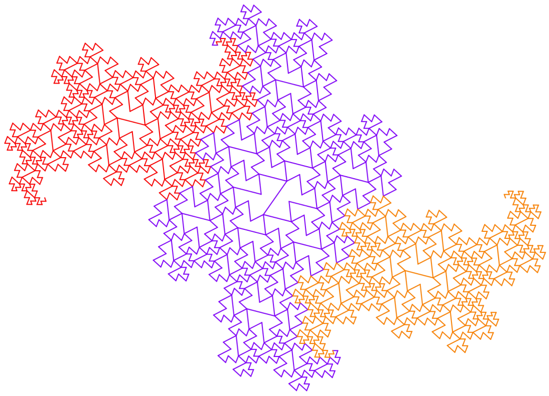
At step 3 we see for the first time that the fractal touches itself. In the enclosed form you can see a duck with some imagination, and point-symmetrically another underneath. That is why I call this fractal the **Duck curve** and the enclosed form the **Duck figure**.


step 4

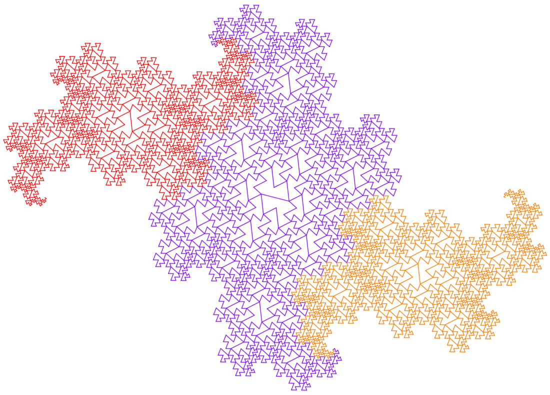

step 5



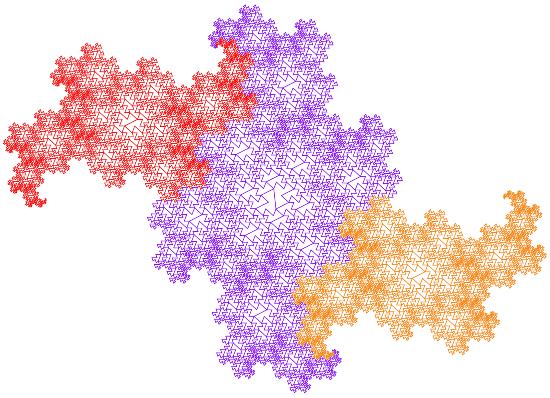
step 6



step 7



step 8

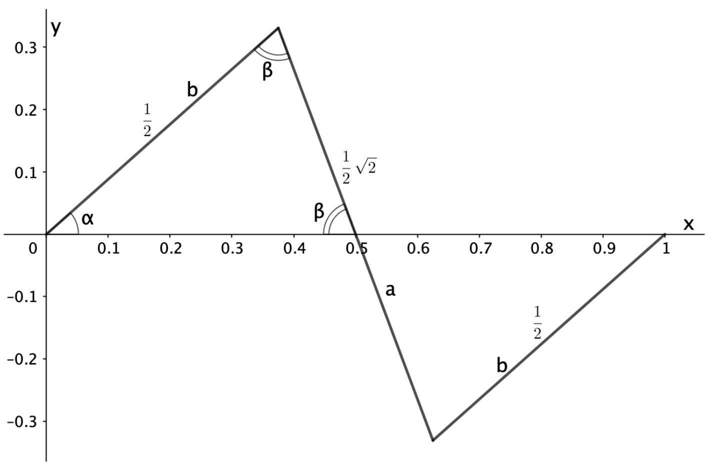


step 9

The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.

The Duck figure that arises in step 3 is seen in step 4 in two smaller sizes, namely a factor $\sqrt{2}$ and 2 smaller. In step 5 there are three formats, which are a factor 2, $2\sqrt{2}$ and 4 smaller than the Duck figure in step 3. With each next step, the number of sizes is increased by 1 and the Duck figures become smaller and smaller.
It is also special that new Duck figures are created at the transitions from one color to another.

Length and direction of line segments

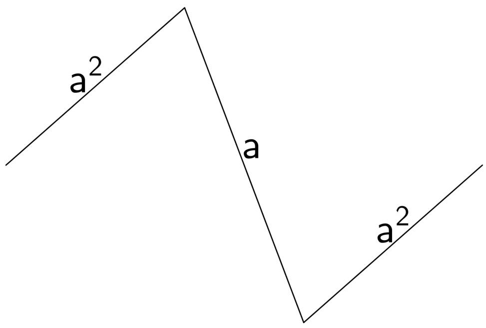


Let's take another look at step 1 of the Duck curve. The part above the *x*-axis, together with the *x*-axis, forms an isosceles triangle. So:

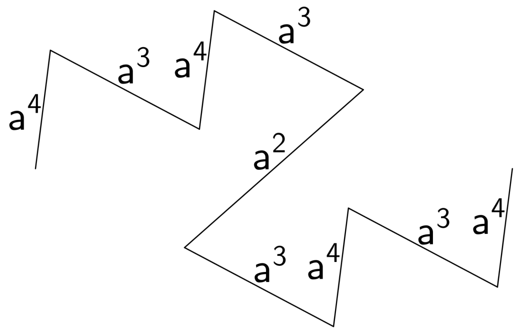
$\alpha + 2\beta = 180° \rightarrow \beta = 90° - \frac{1}{2}\alpha.$

Applying this step 1 therefore gives with line segment *b*: a

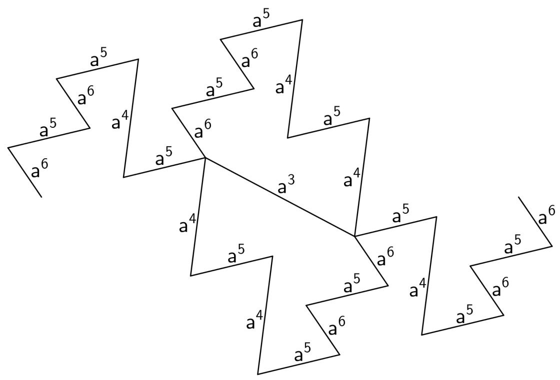
rotation over an angle α and a reduction by factor $\frac{1}{2}$. With line segment *a* , this gives a twist over an angle $-\beta$ and a reduction by factor $\frac{1}{2}\sqrt{2}$. When the latter is applied again, the rotation over an angle $-2\beta = -2\left(90° - \frac{1}{2}\alpha\right) = -180° + \alpha$ and a reduction by factor $\left(\frac{1}{2}\sqrt{2}\right)^2 = \frac{1}{2}$. This gives a parallel line segment and reduction as with one use of line segment *b*.
This means that we can write the application of line segment *b* as a^2. Again applying line segment *a* is written as a^3, etc.
This is illustrated in the three steps below.



step 1



step 2



step 3

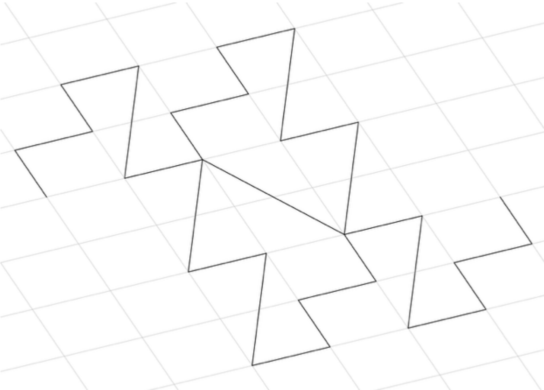
Each line segment in a step is replaced by three line segments in the next step, increasing the exponent of the *a* by 2, 1 and 2. In step *n*, the exponents vary from *n* to 2*n*, where line segments with equal exponents have equal length and are parallel to each other.
Conclusion: across all steps, all line segments of a certain length are parallel to each other. This also means that all Duck figures of a certain size are equally oriented, or rotated 180°.

Besides: $\alpha = \tan^{-1}\left(\frac{1}{3}\sqrt{7}\right) \approx 41,4096°$

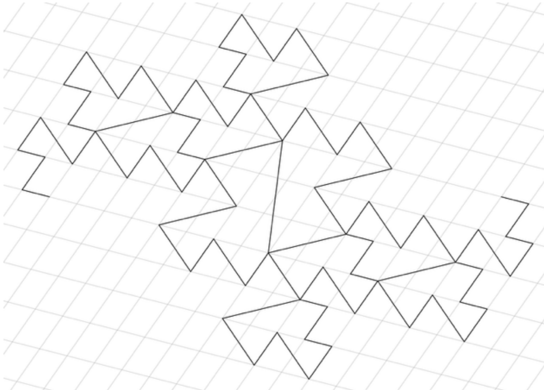
Grid

Herman Haverkot shows that on which grid of parallelograms the Duck curve is based and how that grid changes with each step. Haverkot also gives formal proof that the Duck curve is plane-filling.

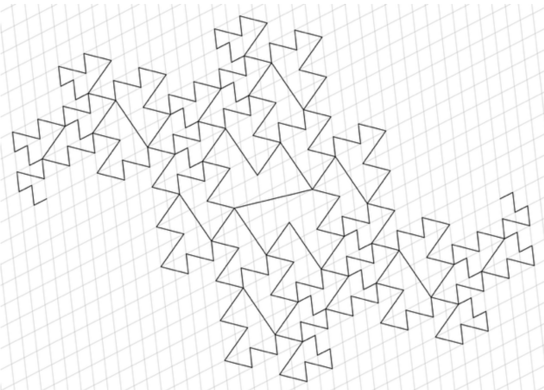
Here are the grids of a number of steps drawn.



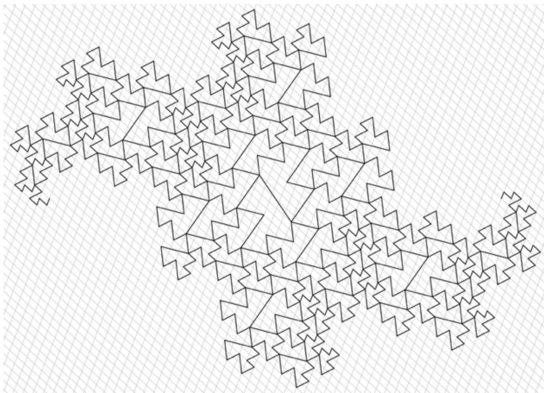
step 3



step 4



step 5

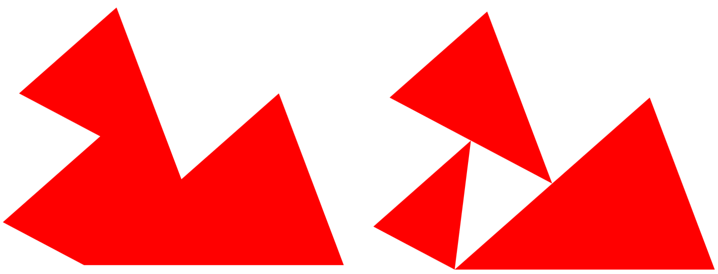


step 6

The parallelograms have sides whose lengths have a ratio of $1 : \sqrt{2}$. The short diagonal is the same length as the long side.
With each step, the grid rotates over an angle α and the parallelograms are halved.

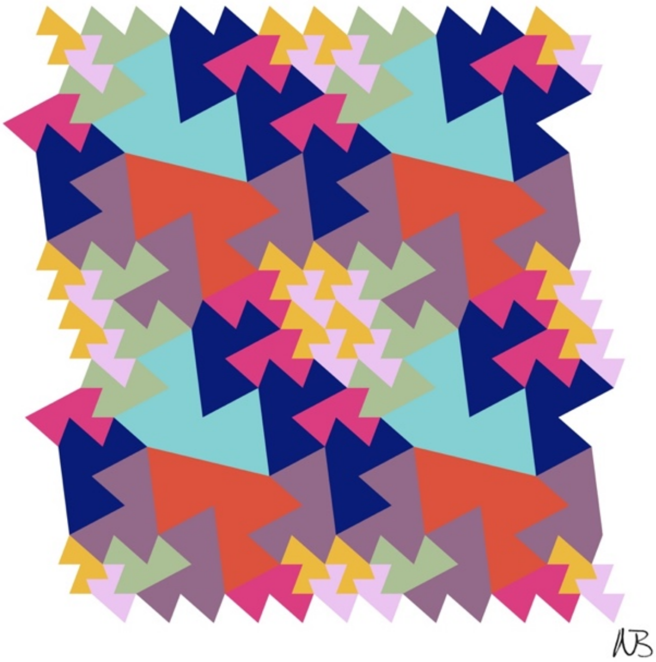
Duck figure

The Duck figure has a number of special properties.
First of all, it can be seen on the right that this figure is made up of four isosceles triangles that are also uniform.
The proportions between the sides of the triangles $\sqrt{2}$, $\sqrt{2}$ and 1.



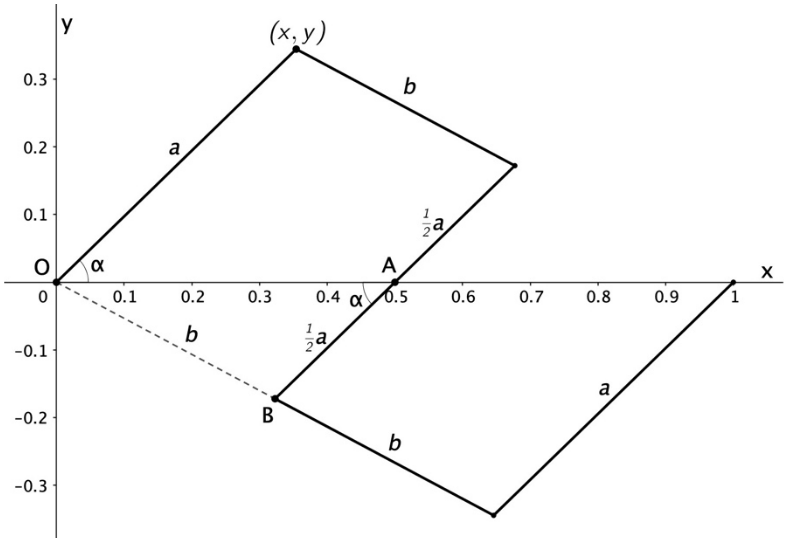
The axis of symmetry of the medium triangle is perpendicular to the middle of the long side of the large triangle; The same goes for the small red and medium triangle. The enclosed white triangle is congruent with the small red triangle.
Because the sizes of the large, medium and small triangles have a ratio of 2, $\sqrt{2}$ and 1, the short side of the large triangle is the same length as the long side of the medium triangle. In addition, the semi-long side of the large triangle is the same length as the short side of the medium-sized triangle. Moreover, these equally long stretches run parallel to each other. The same applies, of course, to the medium and small red triangle.

Due to the special properties of the Duck figure, different sizes can be connected to each other in various ways. This gives all kinds of possibilities to construct tessellations. Below is an example with four formats.



Bird's nests

Another possibility is to make a variant of the 5-Dragon. The 5-Dragon consists of 5 line segments with a length of $\frac{1}{5}\sqrt{5}$. Because of the symmetry, three line segments with length *a* and two with length *b* are now chosen. See figure below.



For the dimension to be 2,:
$3a^2+2b^2=1$.
We again choose the ratio *k* between *a* and *b*: $a=kb$.

With this we express *a* and *b* in *k*:

$$3(kb)^2+2b^2=1 \quad \rightarrow \quad 3k^2b^2+2b^2=1 \quad \rightarrow \quad b^2(3k^2+2)=1$$

$$b^2=\frac{1}{3k^2+2} \quad \rightarrow \quad b=\frac{1}{\sqrt{3k^2+2}}$$

$$a=kb=\frac{k}{\sqrt{3k^2+2}}$$

Applying the cosine rule in $\triangle OAB$ gives:

$$b^2=(\tfrac{1}{2})^2+(\tfrac{1}{2}a)^2-2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}a\cdot\cos\alpha$$

$$\frac{1}{3k^2+2}=\frac{1}{4}+\frac{k^2}{4(3k^2+2)}-\frac{k}{2\sqrt{3k^2+2}}\cos\alpha$$

$$\frac{k}{2\sqrt{3k^2+2}}\cos\alpha=\frac{1}{4}+\frac{k^2}{4(3k^2+2)}-\frac{1}{3k^2+2}$$

$$\cos\alpha=\left[\frac{3k^2+2}{4(3k^2+2)}+\frac{k^2}{4(3k^2+2)}-\frac{4}{4(3k^2+2)}\right]\frac{2\sqrt{3k^2+2}}{k}$$

$$=\frac{4k^2-2}{4(3k^2+2)}\cdot\frac{2\sqrt{3k^2+2}}{k}=\frac{2k^2-1}{k\sqrt{3k^2+2}}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{2k^2-1}{k\sqrt{3k^2+2}}\right)^2}=\sqrt{1-\frac{(2k^2-1)^2}{k^2(3k^2+2)}}$$

$$=\sqrt{\frac{k^2(3k^2+2)}{k^2(3k^2+2)}-\frac{4k^4-4k^2+1}{k^2(3k^2+2)}}=\sqrt{\frac{3k^4+2k^2-4k^4+4k^2-1}{k^2(3k^2+2)}}=\frac{\sqrt{-k^4+6k^2-1}}{k\sqrt{3k^2+2}}$$

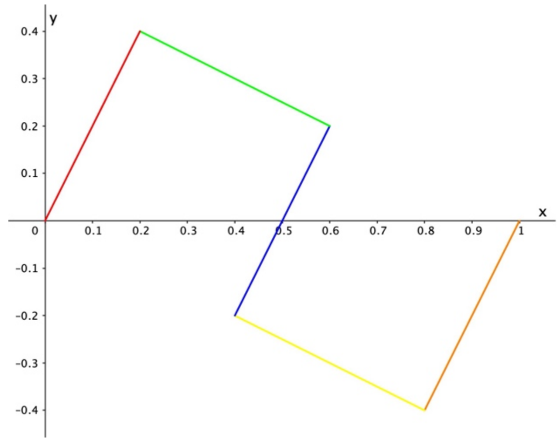
$$x = a \cdot \cos\alpha = \frac{k}{\sqrt{3k^2+2}} \cdot \frac{2k^2-1}{k\sqrt{3k^2+2}} = \frac{2k^2-1}{3k^2+2}$$

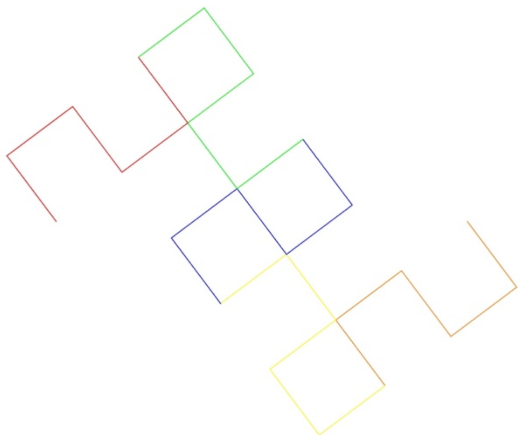
$$y = a \cdot \sin\alpha = \frac{k}{\sqrt{3k^2+2}} \cdot \frac{\sqrt{-k^4+6k^2-1}}{k\sqrt{3k^2+2}} = \frac{\sqrt{-k^4+6k^2-1}}{3k^2+2}$$

We now fill in some values for k.

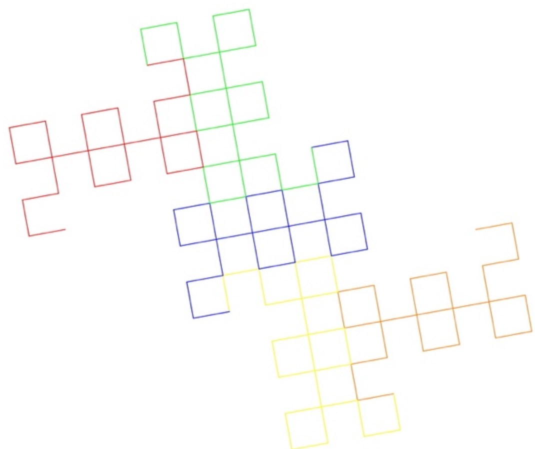
$k = 1$

With $k = 1$ we get an already known fractal, namely the 5-Dragon, of which we have already seen that it is plane-filling.





step 2



step 3

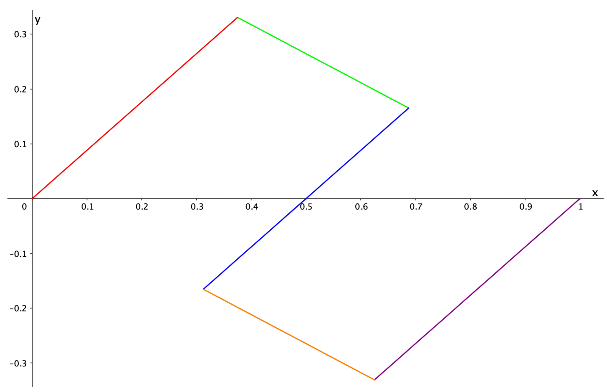
$k = \sqrt{2}$

With $k = \sqrt{2}$ we get:

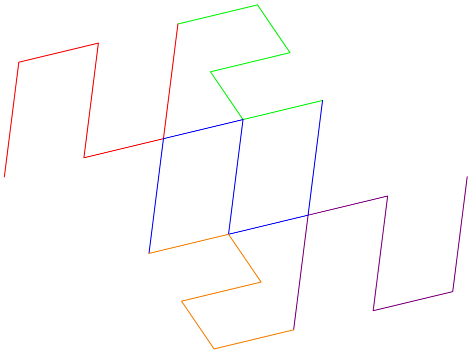
$$a = \frac{\sqrt{2}}{\sqrt{3 \cdot 2 + 2}} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

$$b = \frac{1}{\sqrt{3 \cdot 2 + 2}} = \frac{1}{\sqrt{8}} = \frac{1}{4}\sqrt{2}$$

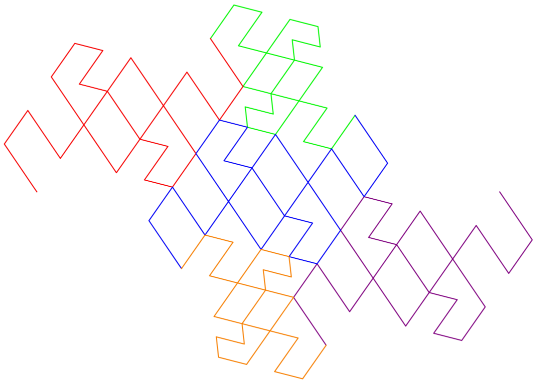
The coordinates of the vertices are respectively: $(0, 0)$, $\left(\frac{3}{8}, \frac{1}{8}\sqrt{7}\right)$, $\left(\frac{11}{16}, \frac{1}{16}\sqrt{7}\right)$, $\left(\frac{5}{16}, -\frac{1}{16}\sqrt{7}\right)$, $\left(\frac{5}{8}, -\frac{1}{8}\sqrt{7}\right)$ and $(1, 0)$.



Step 1 ziet er als hiernaast uit.

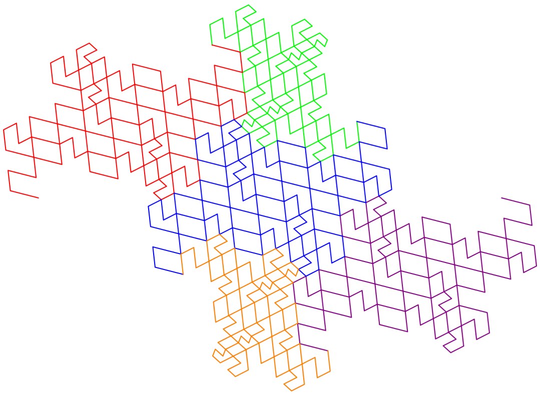


step 2

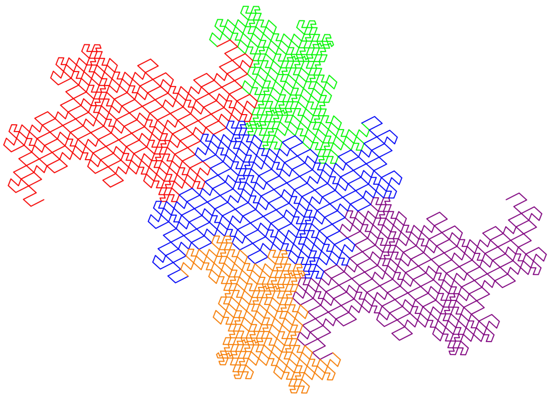


step 3

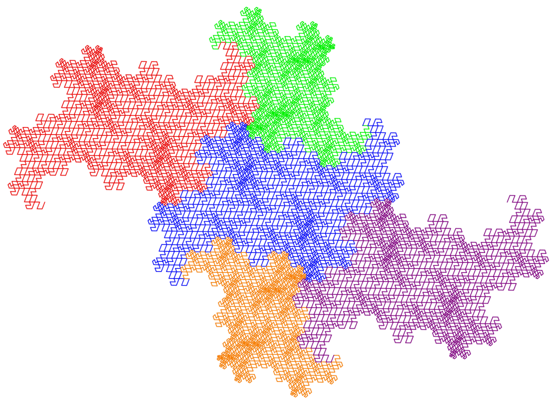
At step 2 we see that the fractal touches itself. In the enclosed form at the top, you can see a bird with some imagination, and the parallelogram below is the nest. That's why I call these fractal **Bird's Nests** .
The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



step 4



step 5



step 6

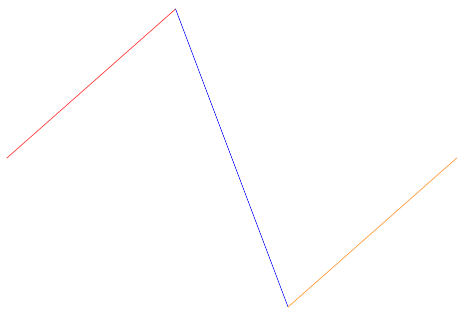
While the Duck curve consists of all similarly shaped figures, this fractal consists of two different figures. In step 2, these two different figures do have the same surface area. The sizes in those successive steps become smaller faster than with the Duck curve. The numbers of figures are also increasing faster.
Across all steps, all line segments of a certain length are parallel to each other. This also means that all figures of a certain size are equally oriented or rotated 180°.

Equal contours

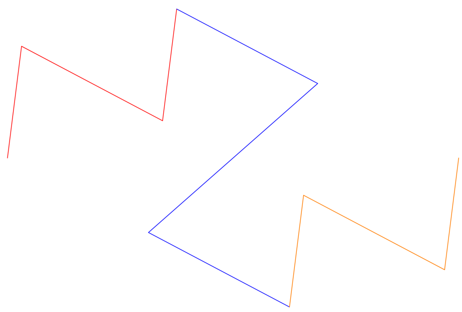
What is striking about step 9 of the Duck curve and step 6 of the Bird's Nest fractal, is that the contours are very similar. Let's take a closer look.

The coordinates of the two vertices of step 1 of the Duck curve are: $\left(\frac{3}{8}, \frac{1}{8}\sqrt{7}\right)$ and $\left(\frac{5}{8}, -\frac{1}{8}\sqrt{7}\right)$. And the coordinates of the four vertices of step 1 of the Bird's Nest fractal are: $\left(\frac{3}{8}, \frac{1}{8}\sqrt{7}\right)$, $\left(\frac{11}{16}, \frac{1}{16}\sqrt{7}\right)$, $\left(\frac{5}{16}, -\frac{1}{16}\sqrt{7}\right)$ and $\left(\frac{5}{8}, -\frac{1}{8}\sqrt{7}\right)$. The first and the last are equal to each other!

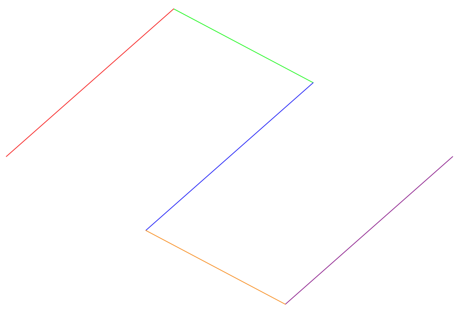
Below are steps 1 and 2 of the Duck curve and step 1 of the Bird's Nest fractal depicted again. The latter is actually a combination of the first two. The left and right line segments of step 1 and the middle three line segments of step 2 of the Duck curve can be seen in step 1 of the Bird's Nest fractal.



Duck curve, step 1

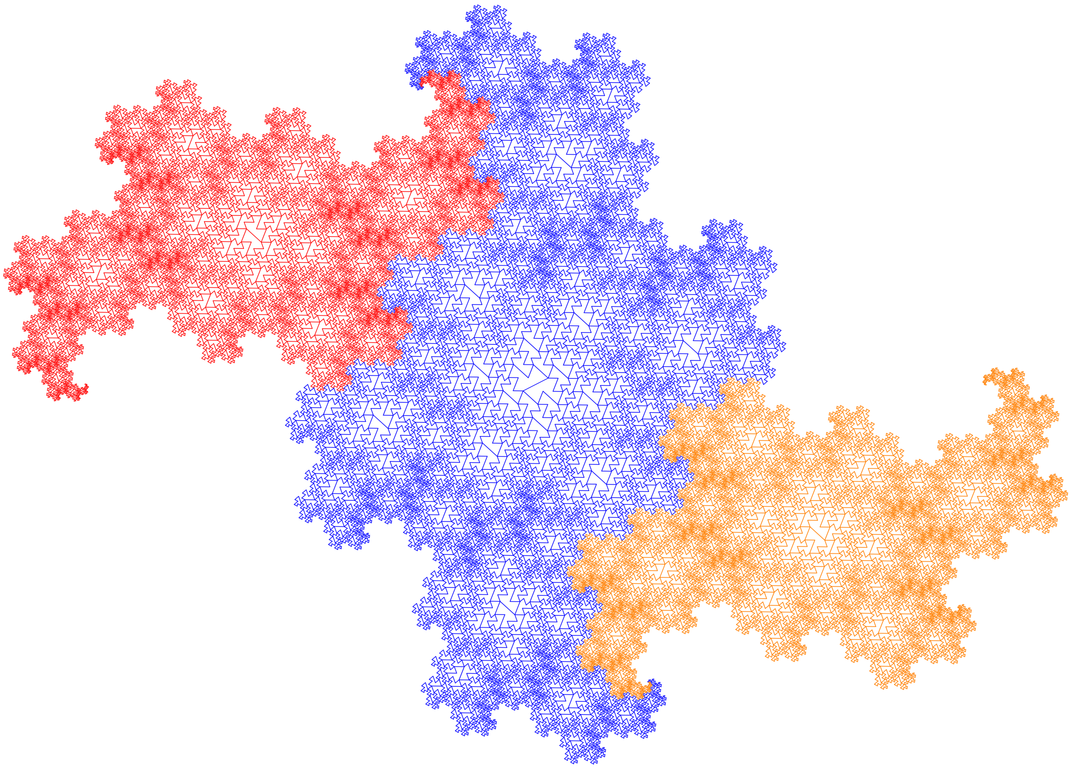


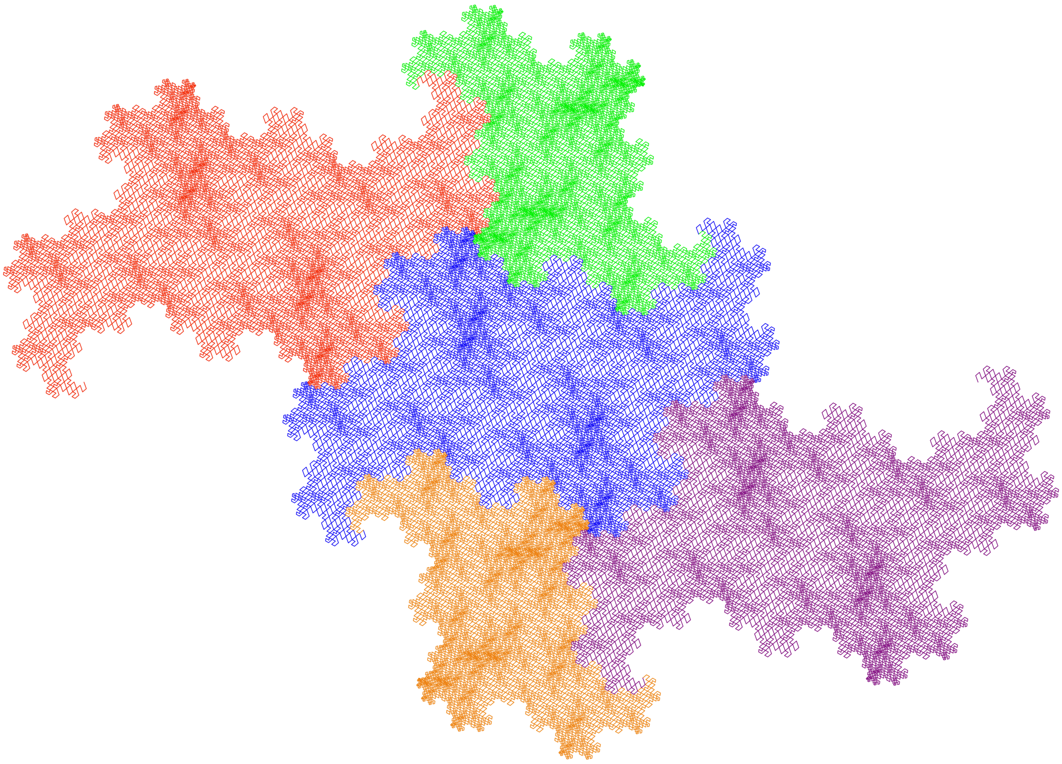
Duck curve, step 2



Bird's Nest fractal, step 1

What actually happens is that the longest line segment in step 1 of the Duck curve is replaced by that same step 1. This means that the Bird's Nest fractal is nothing more than a variant of the Duck curve. And that is why the contours of both fractals look more and more similar with further steps.





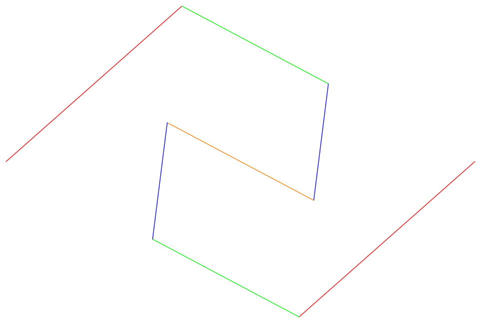
Variants of the Duck curve

It has been shown above that a variant of the Duck curve can be made by replacing a line segment in step 1 with a step 1 of the Duck curve. The resulting fractal is again plane-filling, and the final contours and area of the new fractal are the same as those of the Duck curve. This idea can be taken further.

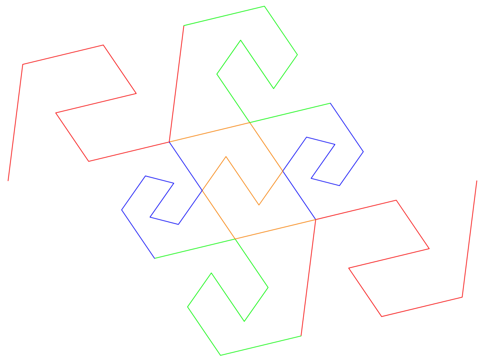
Spirals

By repeatedly replacing the middle line segment in step 1 with step 1 of the Duck curve, increasingly advanced spiral shapes appear.
Here are three examples.

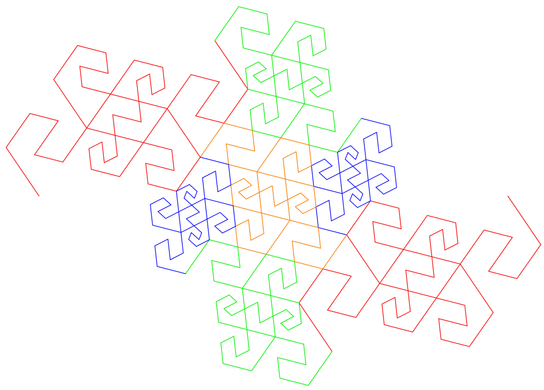
With seven line segments:



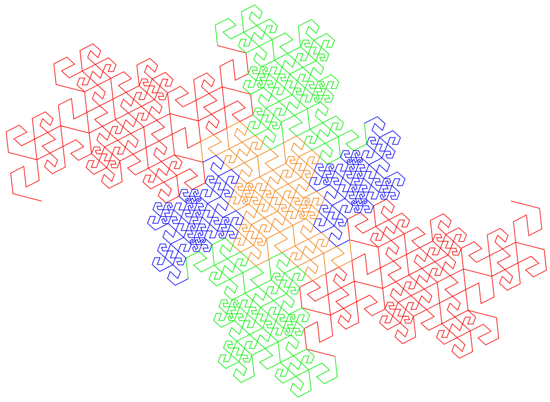
step 1



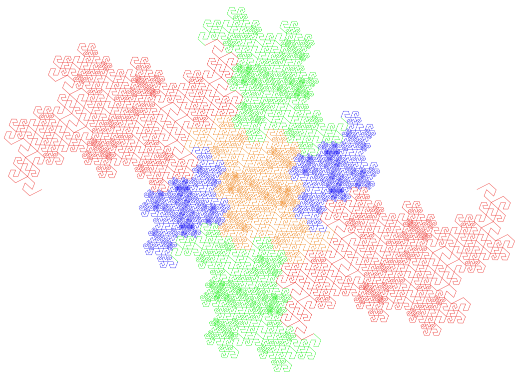
step 2



step 3

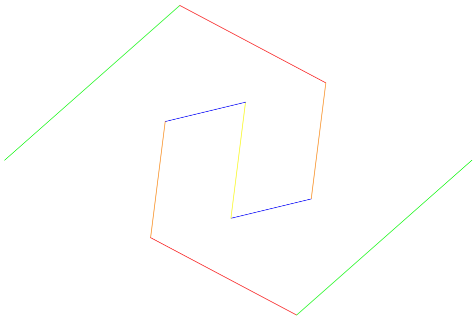


step 4

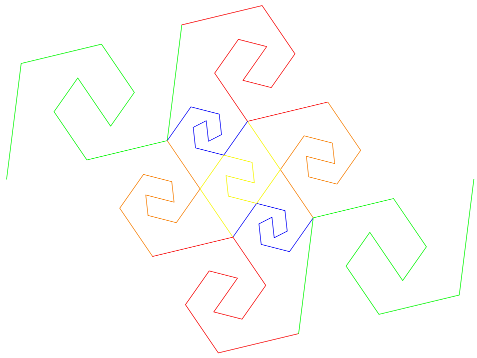


step 5

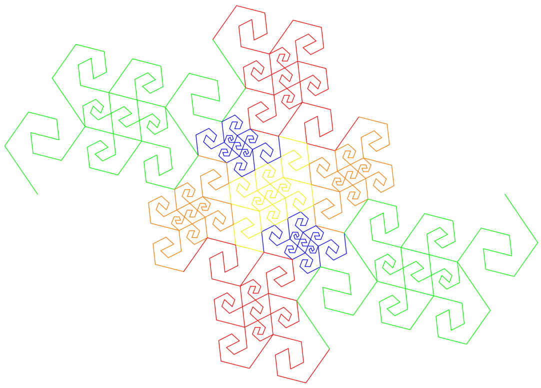
With nine line segments:



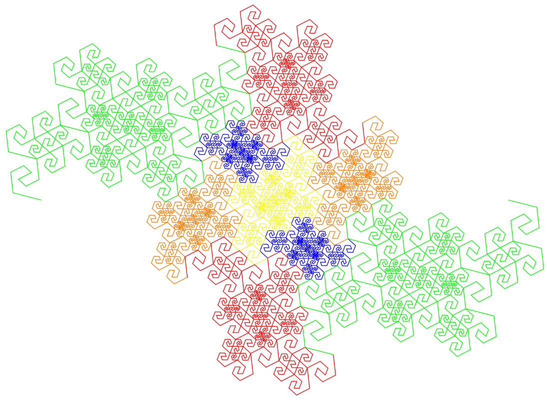
step 1



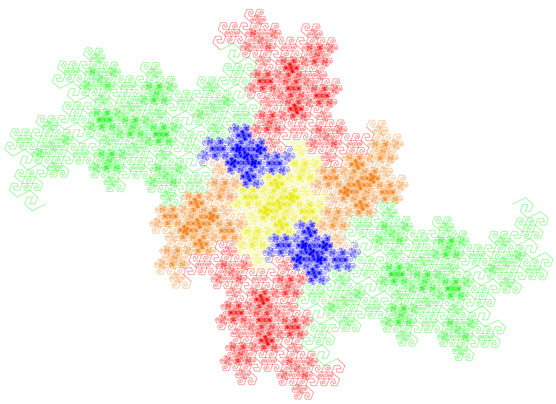
step 2



step 3

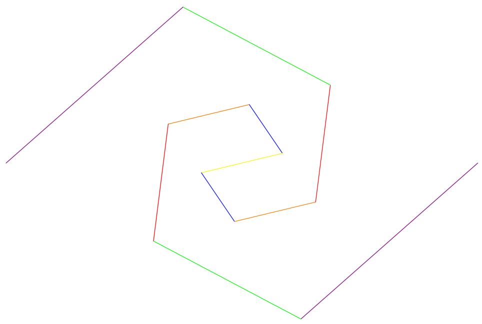


step 4

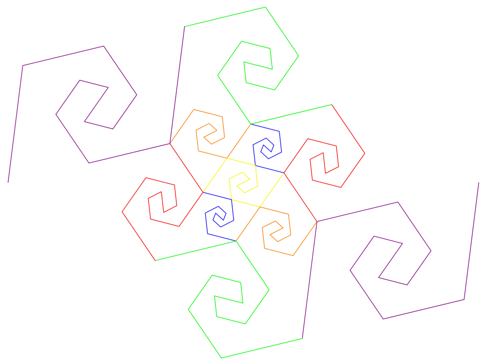


step 5

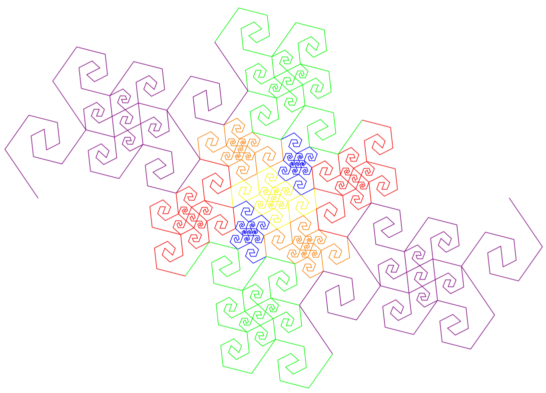
With eleven line segments:



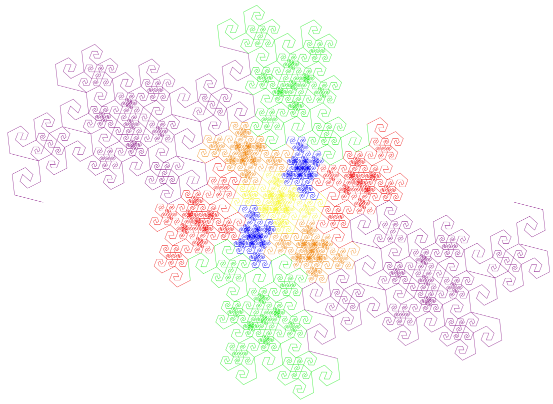
step 1



step 2



step 3



step 4

In principle, this can be implemented indefinitely.

Other examples

Can line segments other than the middle one also be replaced by a step 1 of the Duck curve? The answer is: yes, provided

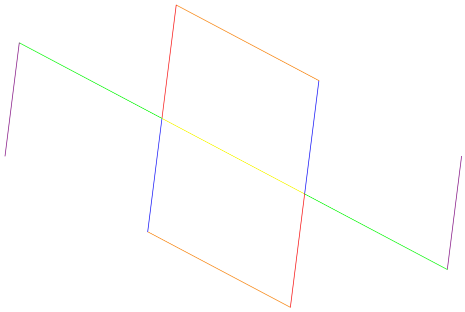
1. only a local longest line segment is replaced, and
2. the replacement – outside the middle line segment – always takes place in duplicate, namely point symmetrical with respect to the center of the fractal.

These two conditions mean that all adjacent line segments in step 1 differ from each other by a factor of $\sqrt{2}$.

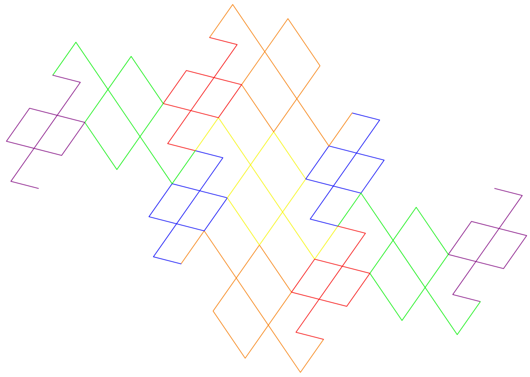
Below are two examples.

With eleven line segments:

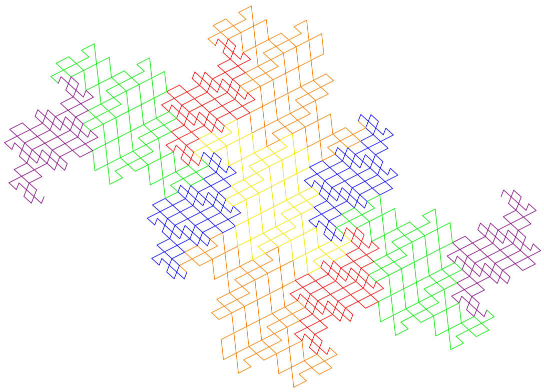
The three longest line segments of the Bird's Nest fractal were replaced in step 1.



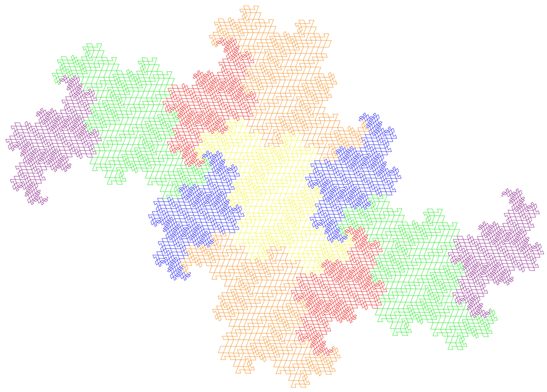
step 1



step 2



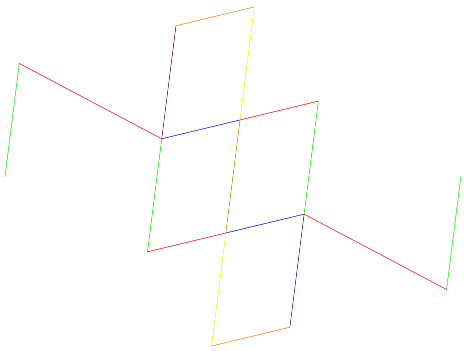
step 3



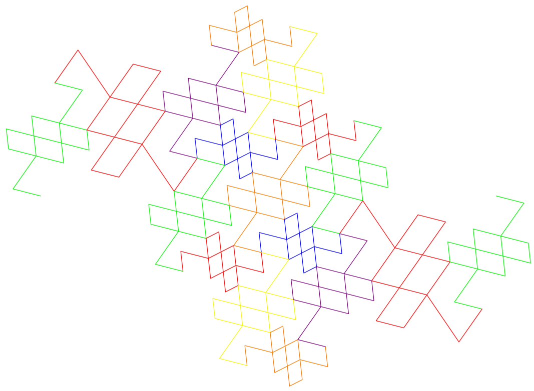
step 4

With seventeen line segments:

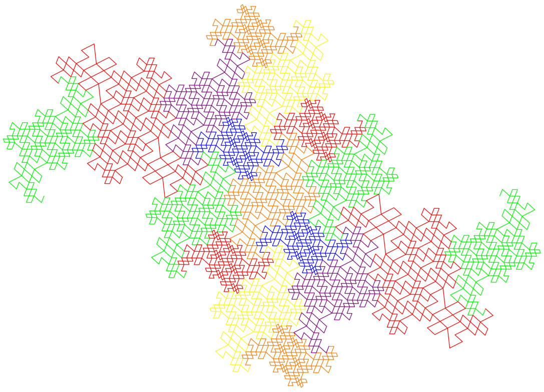
The orange and yellow line segments of the above fractal were replaced in step 1.



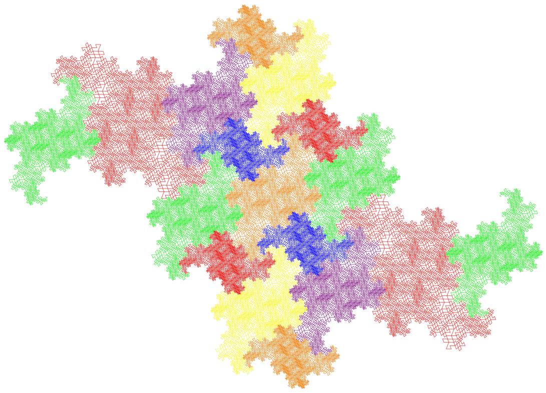
step 1



step 2



step 3



step 4

Of course, there are an infinite number of variants in this way.

A number of things stand out in all the above variants:

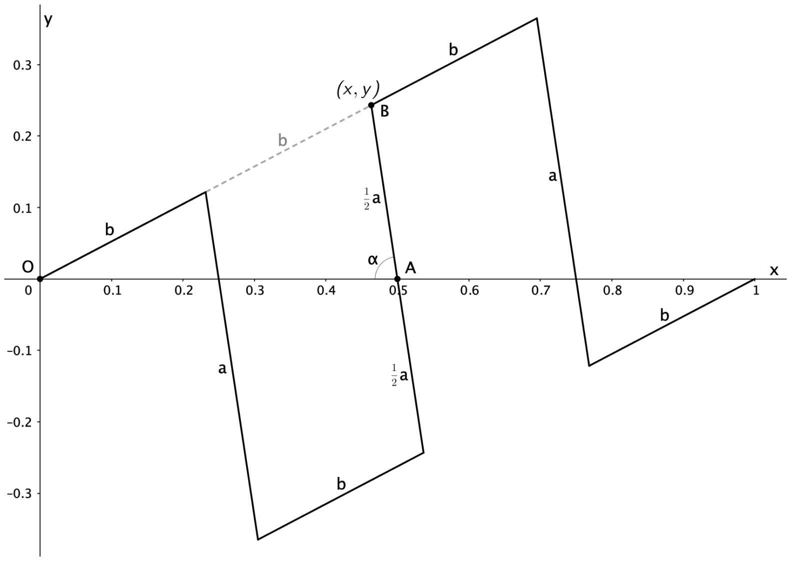
- Over all variants and over all steps thereof, line segments of equal length are also parallel to each other.
- In addition to the Duck figure, the bird and its nest (parallelogram), more and more different enclosed forms are emerging.
- The final contours of all variants are the same as those of the Duck curve.
- The lengths of the line segments in step 1 differ from each other by one or more factors $\sqrt{2}$. The corresponding colored subsurfaces in the final fractal differ from each other by the same number of factors 2.

Titanic

In addition to the Duck curve and its many variants, there is another series of fractals, all of which are not tied to a triangular or square grid.

Titanic2

After the Duck curve with three line segments in step 1 and the Bird's Nest fractal with five line segments, we will now look at a fractal with seven line segments. Because of symmetry and simplicity of calculations, we choose the design on the right, with two different lengths *a* and *b* for the successive line segments.



For the dimension to be 2,
$3a^2+4b^2=1$.
We again choose the ratio *k* between *a* and *b*: $a=kb$.
With this we express *a* and *b* in *k*:

$$3(kb)^2+4b^2=1 \quad\rightarrow\quad 3k^2b^2+4b^2=1 \quad\rightarrow\quad b^2(3k^2+4)=1$$

$$b^2=\frac{1}{3k^2+4} \quad\rightarrow\quad b=\frac{1}{\sqrt{3k^2+4}}$$

$$a=kb=\frac{k}{\sqrt{3k^2+4}}$$

The cosine rule in $\triangle OAB$ gives:

$$(2b)^2=(\tfrac{1}{2})^2+(\tfrac{1}{2}a)^2-2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}a\cdot\cos\alpha$$

$$\frac{4}{3k^2+4}=\tfrac{1}{4}+\frac{k^2}{4(3k^2+4)}-\frac{k}{2\sqrt{3k^2+4}}\cos\alpha$$

$$\frac{k}{2\sqrt{3k^2+4}}\cos\alpha=\tfrac{1}{4}+\frac{k^2}{4(3k^2+4)}-\frac{4}{3k^2+4}$$

$$\cos\alpha=\left[\frac{3k^2+4}{4(3k^2+4)}+\frac{k^2}{4(3k^2+4)}-\frac{16}{4(3k^2+4)}\right]\frac{2\sqrt{3k^2+4}}{k}$$

$$=\frac{4k^2-12}{4(3k^2+4)}\cdot\frac{2\sqrt{3k^2+4}}{k}=\frac{2k^2-6}{k\sqrt{3k^2+4}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{2k^2-6}{k\sqrt{3k^2+4}}\right)^2} = \sqrt{1-\frac{(2k^2-6)^2}{k^2(3k^2+4)}}$$

$$= \sqrt{\frac{k^2(3k^2+4)}{k^2(3k^2+4)} - \frac{4k^4-24k^2+36}{k^2(3k^2+4)}} = \sqrt{\frac{3k^4+4k^2-4k^4+24k^2-36}{k^2(3k^2+4)}}$$

$$= \frac{\sqrt{-k^4+28k^2-36}}{k\sqrt{3k^2+4}}$$

$$x = \tfrac{1}{2} - \tfrac{1}{2}a\cdot\cos\alpha = \tfrac{1}{2} - \frac{k}{2\sqrt{3k^2+4}} \cdot \frac{2k^2-6}{k\sqrt{3k^2+4}} = \tfrac{1}{2} - \frac{2k^2-6}{2(3k^2+4)}$$

$$= \frac{3k^2+4}{2(3k^2+4)} - \frac{2k^2-6}{2(3k^2+4)} = \frac{k^2+10}{2(3k^2+4)}$$

$$y = \tfrac{1}{2}a\cdot\sin\alpha = \frac{k}{2\sqrt{3k^2+4}} \cdot \frac{\sqrt{-k^4+28k^2-36}}{k\sqrt{3k^2+4}} = \frac{\sqrt{-k^4+28k^2-36}}{2(3k^2+4)}$$

$k = 2$

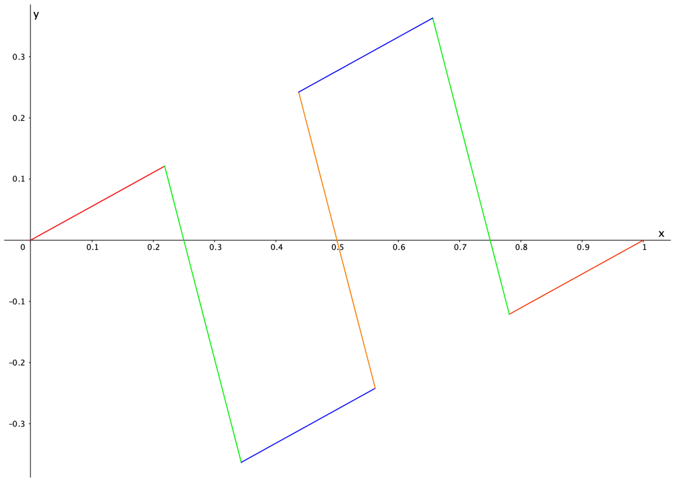
For $k = 2$ we get:

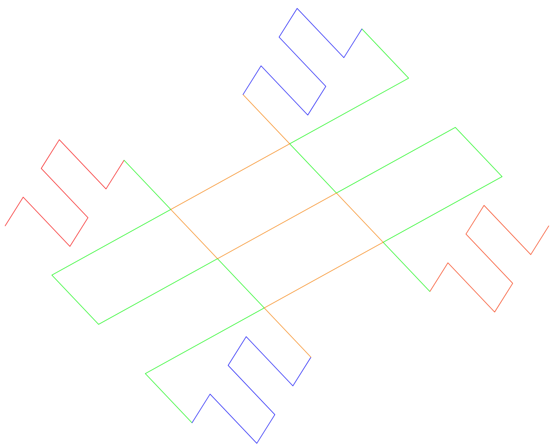
$$a = \frac{2}{\sqrt{3\cdot 2^2+4}} = \frac{1}{2}$$

$$b = \frac{1}{\sqrt{3\cdot 2^2+4}} = \frac{1}{4}$$

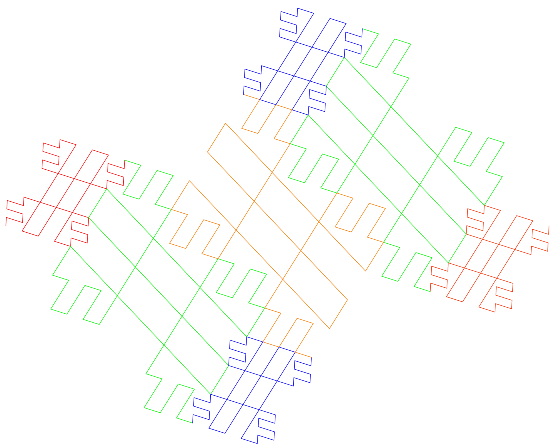
The coordinates of the vertices are respectively: $(0, 0)$, $\left(\frac{7}{32}, \frac{1}{32}\sqrt{15}\right)$, $\left(\frac{11}{32}, -\frac{3}{32}\sqrt{15}\right)$, $\left(\frac{9}{16}, -\frac{1}{16}\sqrt{15}\right)$, $\left(\frac{7}{16}, \frac{1}{16}\sqrt{15}\right)$, $\left(\frac{21}{32}, \frac{3}{32}\sqrt{15}\right)$, $\left(\frac{25}{32}, -\frac{1}{32}\sqrt{15}\right)$ and $(1, 0)$.

Step 1 looks like the one on the right.



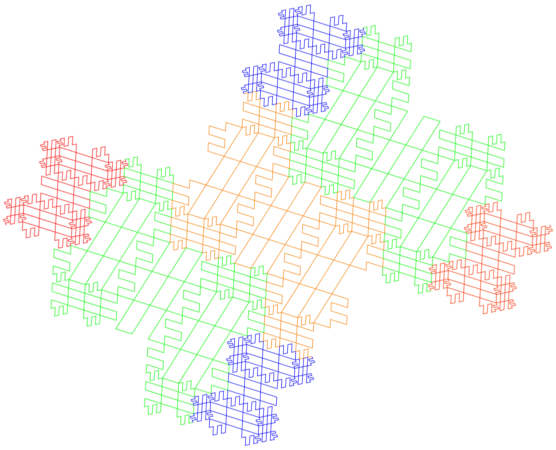


step 2

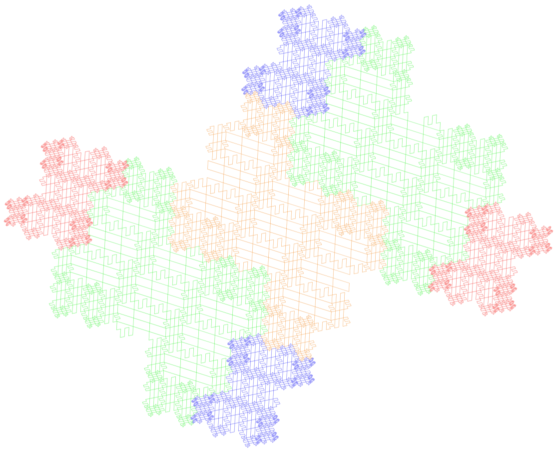


step 3

At step 2 we see that the fractal touches itself. In the enclosed form at the top, you can see a sinking Titanic with some imagination, two funnels of which are still visible. That's why I call this fractal **Titanic2**.
The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



step 4



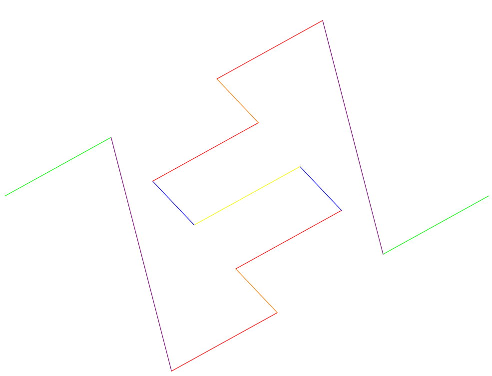
step 5

This fractal is substantially different from the Duck curve: the contours look different and this time the ratio between the lengths of the line segments is not $\sqrt{2}$, but 2.

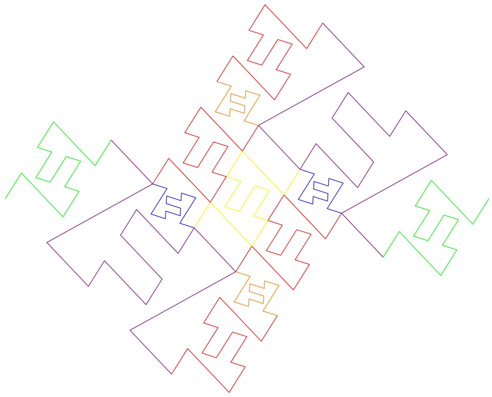
The shapes of the enclosed figures become more and more diverse and from step 4 onwards enclosed figures arise that no longer have a surface area ratio $1:4$ compared to most other enclosed figures. This makes it impossible to calculate the final enclosed area with the help of counts, because it is not possible to predict what other shapes the enclosed figures will take when walking further. We will come back to this later.

Titanic2 variant

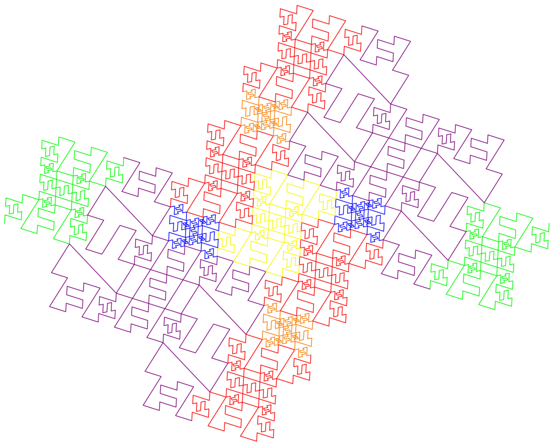
As with the Duck curve, we can see if we can make a variant of the Titanic2 fractal, which is also plane-filling. We replace the middle line segment of step 1 with the same step 1.
In step 4 it can be seen that the contours of this fractal become the same as the contours of the Titanic2 fractal. The final enclosed surfaces will be equal to each other.



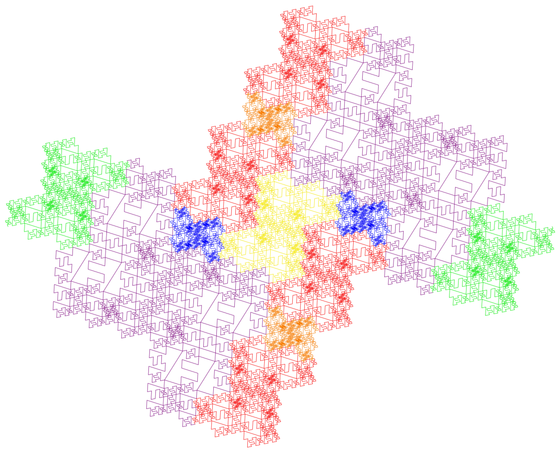
step 1



step 2



step 3



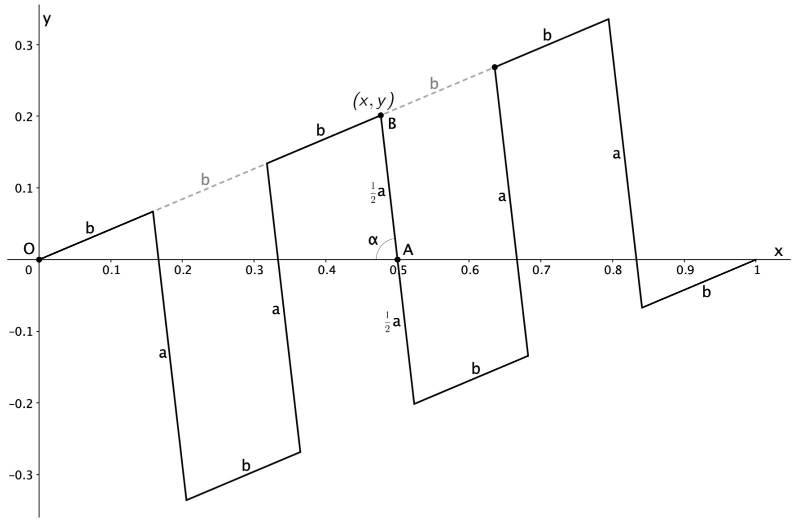
step 4

There are probably many more variants of the Titanic2 fractal to be found, in the same way as the Duck curve. The number of line segments in step 1 of the variants increases rapidly, because each replacement of a line segment gives six extra line segments.
I will limit myself here to this one example.

Titanic3

We are now going to look at a fractal with eleven line segments.
Because of symmetry and simplicity of calculations, we choose the design on the right, with two different lengths *a* and *b* for the successive line segments.

For the dimension to be 2, $5a^2 + 6b^2 = 1$.



We again choose the ratio *k* between *a* and *b*: $a = kb$.
With this we express *a* and *b* in *k*:

$$5(kb)^2 + 6b^2 = 1 \quad \rightarrow \quad 5k^2b^2 + 6b^2 = 1 \quad \rightarrow \quad b^2(5k^2 + 6) = 1$$

$$b^2 = \frac{1}{5k^2+6} \quad \rightarrow \quad b = \frac{1}{\sqrt{5k^2+6}}$$

$$a = kb = \frac{k}{\sqrt{5k^2+6}}$$

The cosine rule in $\triangle OAB$ gives:

$$(3b)^2 = (\tfrac{1}{2})^2 + (\tfrac{1}{2}a)^2 - 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}a \cdot \cos\alpha$$

$$\frac{9}{5k^2+6} = \tfrac{1}{4} + \frac{k^2}{4(5k^2+6)} - \frac{k}{2\sqrt{5k^2+6}}\cos\alpha$$

$$\frac{k}{2\sqrt{5k^2+6}}\cos\alpha = \tfrac{1}{4} + \frac{k^2}{4(5k^2+6)} - \frac{9}{5k^2+6}$$

$$\cos\alpha = \left[\frac{5k^2+6}{4(5k^2+6)} + \frac{k^2}{4(5k^2+6)} - \frac{36}{4(5k^2+6)}\right]\frac{2\sqrt{5k^2+6}}{k}$$

$$= \frac{6k^2-30}{4(5k^2+6)} \cdot \frac{2\sqrt{5k^2+6}}{k} = \frac{3k^2-15}{k\sqrt{5k^2+6}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3k^2-15}{k\sqrt{5k^2+6}}\right)^2} = \sqrt{1-\frac{(3k^2-15)^2}{k^2(5k^2+6)}}$$

$$= \sqrt{\frac{k^2(5k^2+6)}{k^2(5k^2+6)} - \frac{9k^4-90k^2+225}{k^2(5k^2+6)}} = \sqrt{\frac{5k^4+6k^2-9k^4+90k^2-225}{k^2(5k^2+6)}}$$

$$= \frac{\sqrt{-4k^4+96k^2-225}}{k\sqrt{5k^2+6}}$$

$$x = \tfrac{1}{2} - \tfrac{1}{2}a \cdot \cos\alpha = \tfrac{1}{2} - \frac{k}{2\sqrt{5k^2+6}} \cdot \frac{3k^2-15}{k\sqrt{5k^2+6}} = \tfrac{1}{2} - \frac{3k^2-15}{2(5k^2+6)}$$

$$= \frac{5k^2+6}{2(5k^2+6)} - \frac{3k^2-15}{2(5k^2+6)} = \frac{2k^2+21}{2(5k^2+6)}$$

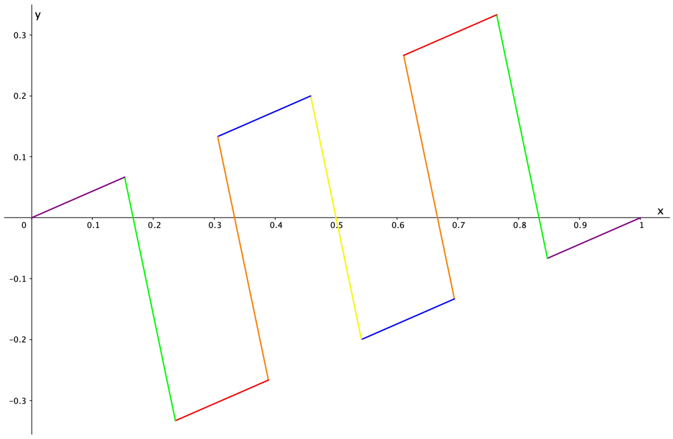
$$y = \tfrac{1}{2}a \cdot \sin\alpha = \frac{k}{2\sqrt{5k^2+6}} \cdot \frac{\sqrt{-4k^4+96k^2-225}}{k\sqrt{5k^2+6}} = \frac{\sqrt{-4k^4+96k^2-225}}{2(5k^2+6)}$$

$k = \sqrt{6}$

For $k = \sqrt{6}$ we get:

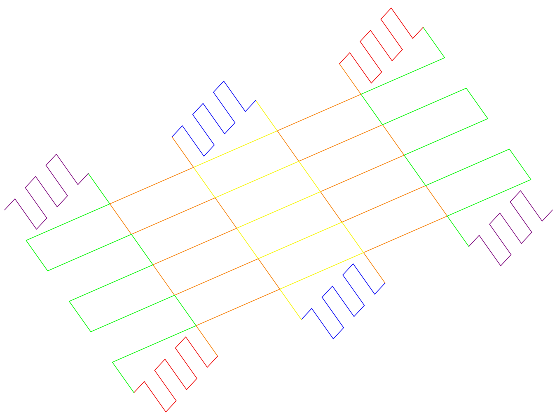
$$a = \frac{\sqrt{6}}{\sqrt{5 \cdot (\sqrt{6})^2 + 6}} = \frac{1}{6}\sqrt{6}$$

$$b = \frac{1}{\sqrt{5 \cdot (\sqrt{6})^2 + 6}} = \frac{1}{6}$$

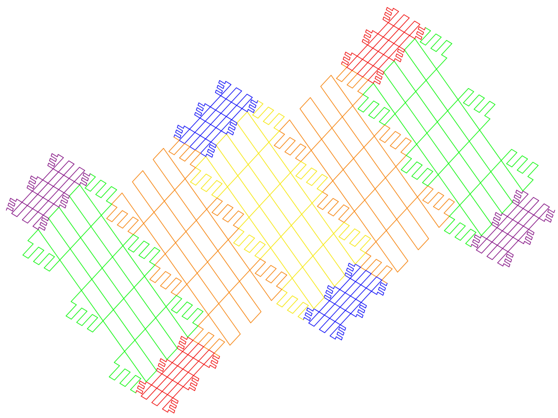


The coordinates of the vertices are resp.: $(0, 0)$, $\left(\frac{11}{72}, \frac{1}{72}\sqrt{23}\right)$, $\left(\frac{17}{72}, -\frac{5}{72}\sqrt{23}\right)$, $\left(\frac{7}{18}, -\frac{1}{18}\sqrt{23}\right)$, $\left(\frac{11}{36}, \frac{1}{36}\sqrt{23}\right)$, $\left(\frac{11}{24}, \frac{1}{24}\sqrt{23}\right)$, $\left(\frac{13}{24}, -\frac{1}{24}\sqrt{23}\right)$, $\left(\frac{25}{36}, -\frac{1}{36}\sqrt{23}\right)$, $\left(\frac{11}{18}, \frac{1}{18}\sqrt{23}\right)$, $\left(\frac{55}{72}, \frac{5}{72}\sqrt{23}\right)$, $\left(\frac{61}{72}, -\frac{1}{72}\sqrt{23}\right)$ and $(1, 0)$.

Step 1 looks like the one on the right.



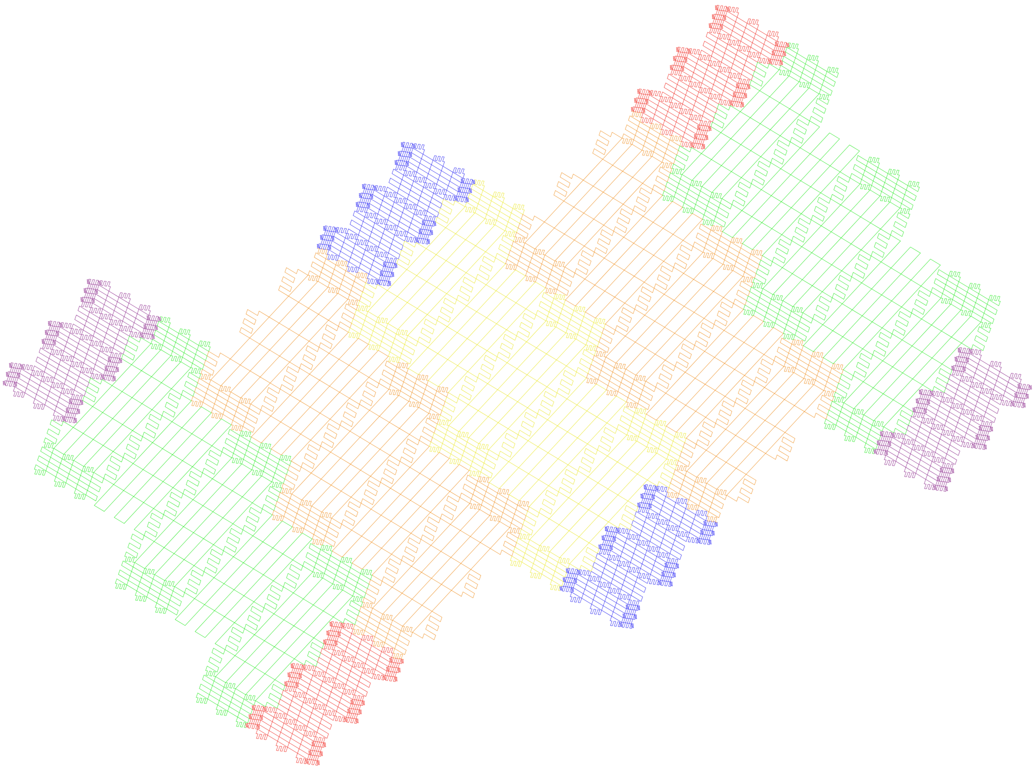
step 2



step 3

At step 2 we see that the fractal touches itself. In the enclosed form at the top, with some imagination, you can see a sinking Titanic, three funnels of which are still visible. That's why I call this fractal **Titanic3**.

The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



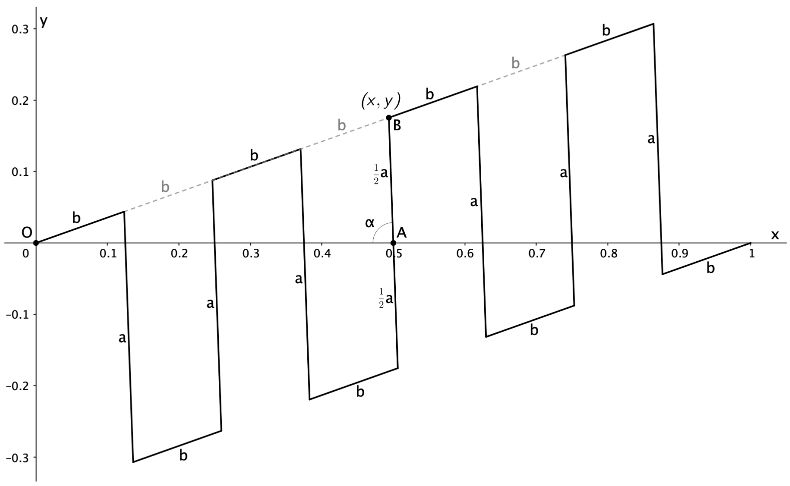
step 4

This fractal is substantially different from the Duck curve and the Titanic2 fractal: the contours look different and this time the ratio between the lengths of the line segments is $\sqrt{6}$.

Titanic4

We are now going to look at a fractal with fifteen line segments.

Because of symmetry and simplicity of calculations, we choose the design on the right, with two different lengths *a* and *b* for the successive line segments.



For the dimension to be 2, $7a^2 + 8b^2 = 1$.

We again choose the ratio *k* between *a* and *b*: $a = kb$.

With this we express *a* and *b* in *k*:

$$7(kb)^2 + 8b^2 = 1 \quad \rightarrow \quad 7k^2b^2 + 8b^2 = 1 \quad \rightarrow \quad b^2(7k^2 + 8) = 1$$

$$b^2 = \frac{1}{7k^2 + 8} \quad \rightarrow \quad b = \frac{1}{\sqrt{7k^2 + 8}}$$

$$a = kb = \frac{k}{\sqrt{7k^2+8}}$$

The cosine rule in $\triangle OAB$ gives:

$$(4b)^2 = (\tfrac{1}{2})^2 + (\tfrac{1}{2}a)^2 - 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} a \cdot \cos\alpha$$

$$\frac{16}{7k^2+8} = \frac{1}{4} + \frac{k^2}{4(7k^2+8)} - \frac{k}{2\sqrt{7k^2+8}}\cos\alpha$$

$$\frac{k}{2\sqrt{7k^2+8}}\cos\alpha = \frac{1}{4} + \frac{k^2}{4(7k^2+8)} - \frac{16}{7k^2+8}$$

$$\cos\alpha = \left[\frac{7k^2+8}{4(7k^2+8)} + \frac{k^2}{4(7k^2+8)} - \frac{64}{4(7k^2+8)}\right]\frac{2\sqrt{7k^2+8}}{k}$$

$$= \frac{8k^2-56}{4(7k^2+8)} \cdot \frac{2\sqrt{7k^2+8}}{k} = \frac{4k^2-28}{k\sqrt{7k^2+8}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{4k^2-28}{k\sqrt{7k^2+8}}\right)^2} = \sqrt{1-\frac{(4k^2-28)^2}{k^2(7k^2+8)}}$$

$$= \sqrt{\frac{k^2(7k^2+8)}{k^2(7k^2+8)} - \frac{16k^4-224k^2+784}{k^2(7k^2+8)}} = \sqrt{\frac{7k^4+8k^2-16k^4+224k^2-784}{k^2(7k^2+8)}}$$

$$= \frac{\sqrt{-9k^4+232k^2-784}}{k\sqrt{7k^2+8}}$$

$$x = \frac{1}{2} - \frac{1}{2}a \cdot \cos\alpha = \frac{1}{2} - \frac{k}{2\sqrt{7k^2+8}} \cdot \frac{4k^2-28}{k\sqrt{7k^2+8}} = \frac{1}{2} - \frac{4k^2-28}{2(7k^2+8)}$$

$$= \frac{7k^2+8}{2(7k^2+8)} - \frac{4k^2-28}{2(7k^2+8)} = \frac{3k^2+36}{2(7k^2+8)}$$

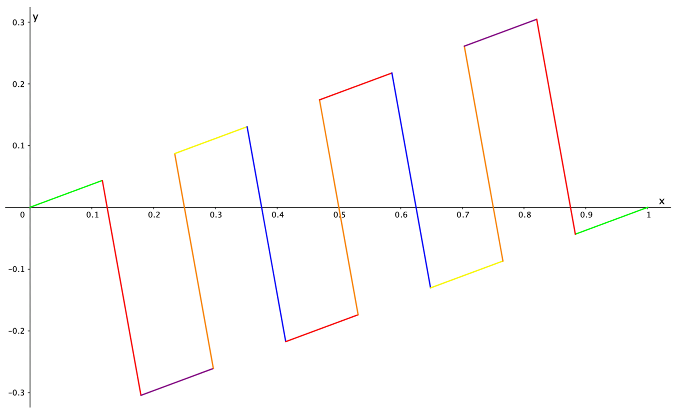
$$y = \frac{1}{2}a \cdot \sin\alpha = \frac{k}{2\sqrt{7k^2+8}} \cdot \frac{\sqrt{-9k^4+232k^2-784}}{k\sqrt{7k^2+8}} = \frac{\sqrt{-9k^4+232k^2-784}}{2(7k^2+8)}$$

$k = \sqrt{8}$

For $k = \sqrt{8}$ we get:

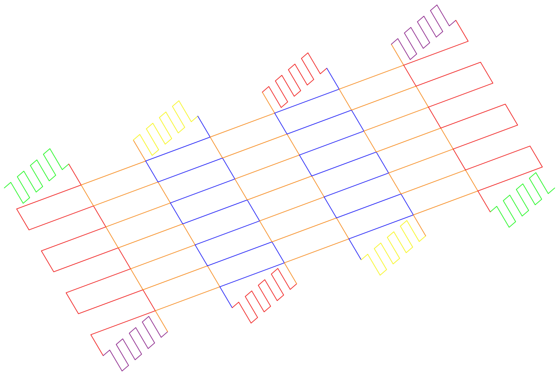
$$a = \frac{\sqrt{8}}{\sqrt{7 \cdot (\sqrt{8})^2 + 8}} = \frac{1}{8}\sqrt{8}$$

$$b = \frac{1}{\sqrt{7 \cdot (\sqrt{8})^2 + 8}} = \frac{1}{8}$$

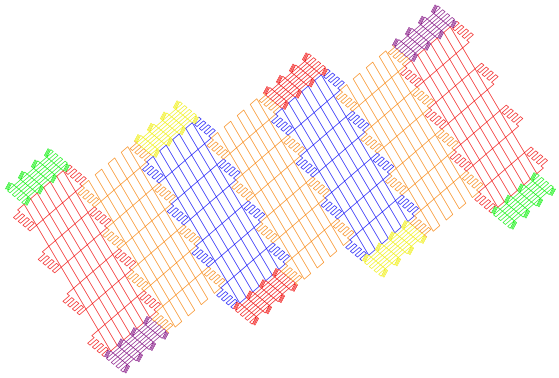


The coordinates of the vertices are resp.: $(0,0)$, $\left(\frac{15}{128}, \frac{1}{128}\sqrt{31}\right)$, $\left(\frac{23}{128}, -\frac{7}{128}\sqrt{31}\right)$, $\left(\frac{19}{64}, -\frac{3}{64}\sqrt{31}\right)$, $\left(\frac{15}{64}, \frac{1}{64}\sqrt{31}\right)$, $\left(\frac{45}{128}, \frac{3}{128}\sqrt{31}\right)$, $\left(\frac{53}{128}, -\frac{5}{128}\sqrt{31}\right)$, $\left(\frac{17}{32}, -\frac{1}{32}\sqrt{31}\right)$, $\left(\frac{15}{32}, \frac{1}{32}\sqrt{31}\right)$, $\left(\frac{75}{128}, \frac{5}{128}\sqrt{31}\right)$, $\left(\frac{83}{128}, -\frac{3}{128}\sqrt{31}\right)$, $\left(\frac{49}{64}, -\frac{1}{64}\sqrt{31}\right)$, $\left(\frac{45}{64}, \frac{3}{64}\sqrt{31}\right)$, $\left(\frac{105}{128}, \frac{7}{128}\sqrt{31}\right)$, $\left(\frac{113}{128}, -\frac{1}{128}\sqrt{31}\right)$ and $(1, 0)$.

Step 1 looks like the one on the right.



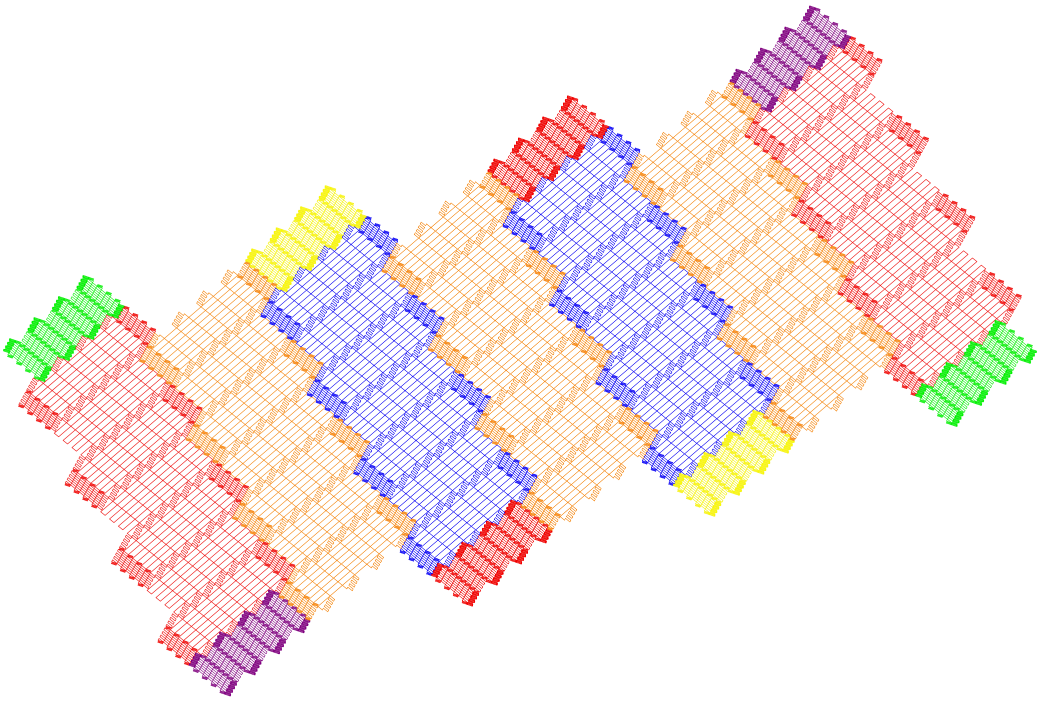
step 2



step 3

At step 2 we see that the fractal touches itself. In the enclosed form at the top, you can see a sinking Titanic with some imagination, with four funnels visible. That's why I call this fractal **Titanic4**.

The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



step 4

This fractal is substantially different from the Duck curve and the other Titanic fractals: the contours look different and this time the ratio between the lengths of the line segments is $\sqrt{8}$.

Similarly, Titanic5, Titanic6, etc. could also be fractalized. Apart from the fact that the Titanic itself had no more than four funnels, the pictures of these fractals are becoming less and less interesting (brick walls?). So, I'll leave it at that.

Enclosed area

I do want to look at the enclosed surface of the Titanic fractals. As noted earlier, it is impossible to predict what new forms the enclosed figures will take with further steps. This makes it impossible to calculate the enclosed area with the help of counts of enclosed figures. That is why we do this via a different method.

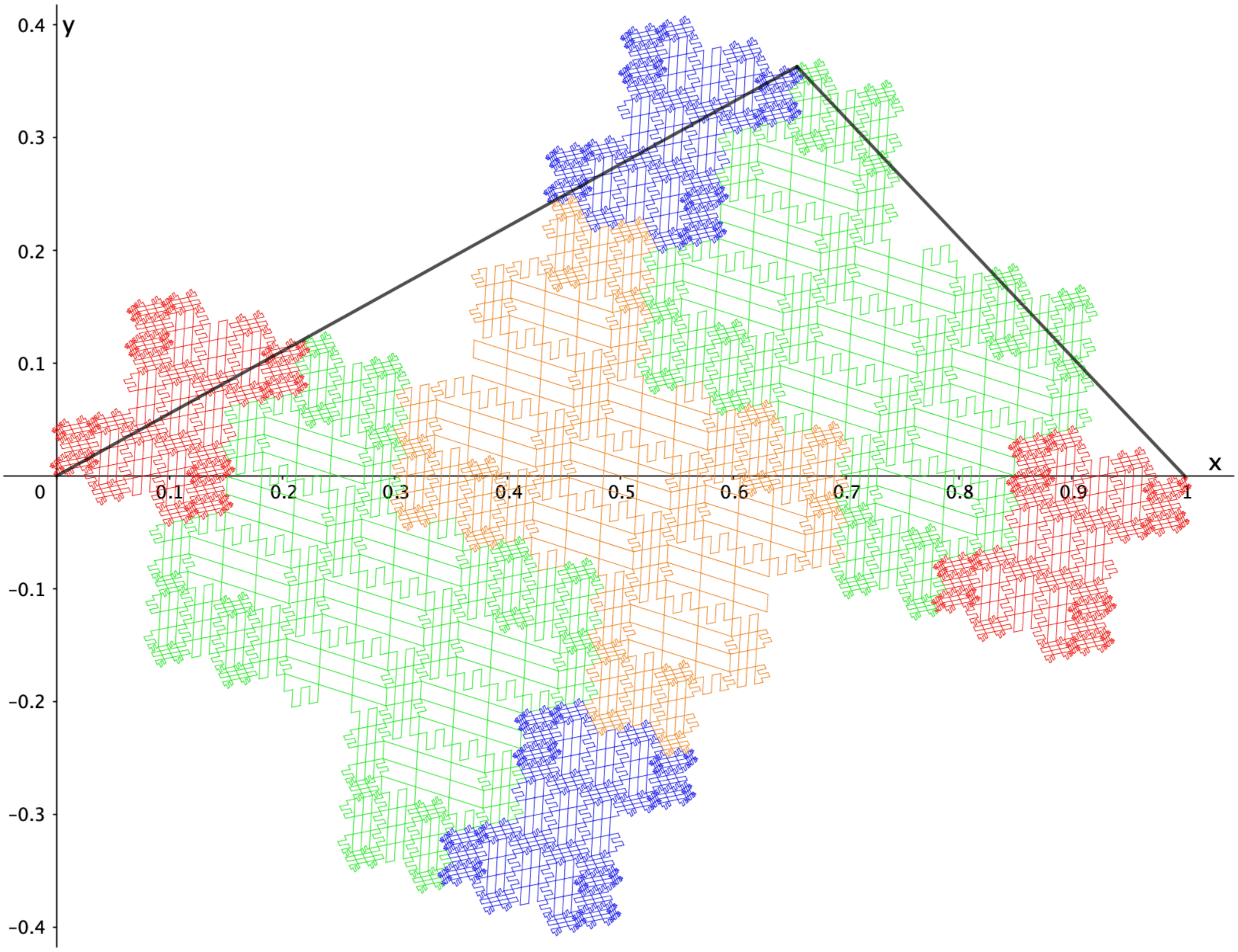
From the Titanic2, Titanic3 and Titanic4 we take a step 5 or 4 and draw two lines from the highest point of step 1 to $(0, 0)$ and to $(1, 0)$. Together with the *x*-axis, triangles are created that roughly correspond to the shape of the fractal above the *x*-axis.

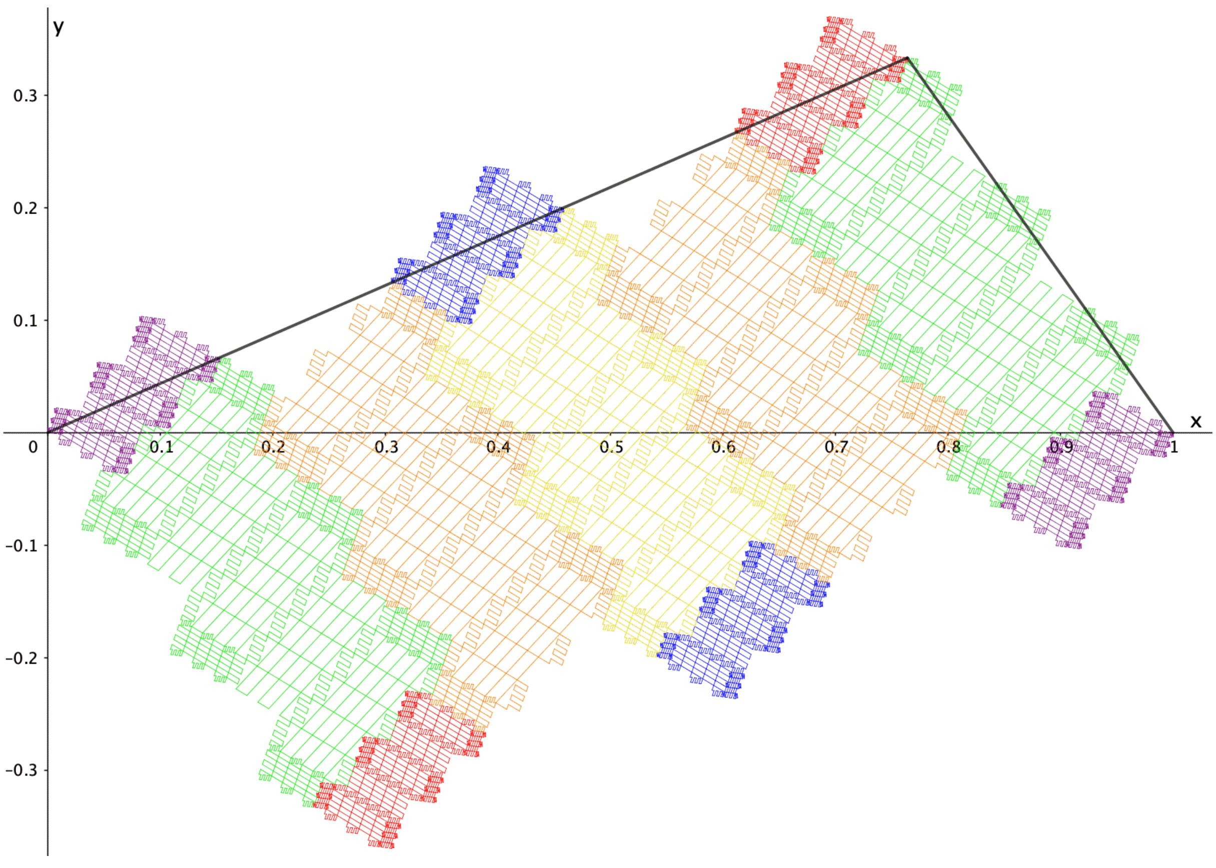
On the right side we see 2, 3 and 4 pieces of the fractal falling outside the triangle, and also 2, 3 and 4 white pieces inside the triangle. If we were to let the steps go to infinity, the areas of the areas inside and outside the triangle would be exactly the same size. For the area determination we can use the straight line of the triangle on the right.

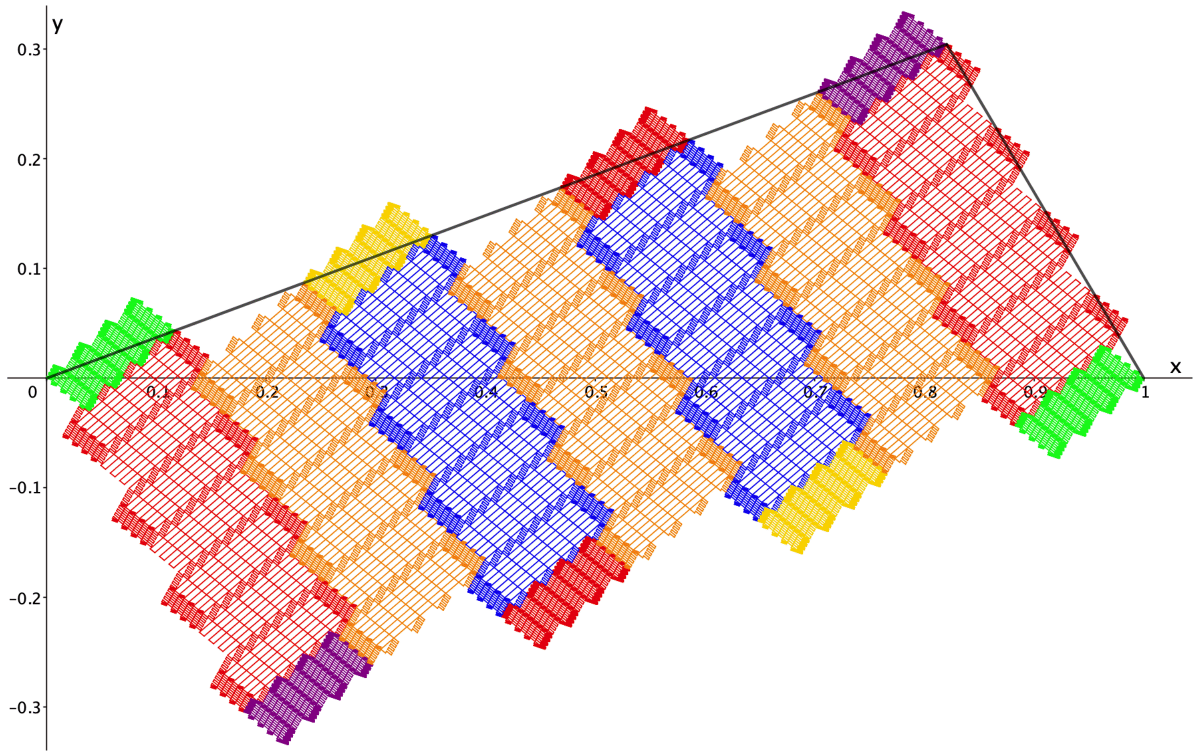
Something similar can be seen on the left side of the triangle. However, there remains an area outside the triangle. So we have to add the area of that area to the area of the triangle.

Now we see that such an area outside the triangle is exactly half of a smaller copy of the entire fractal. We can also approximate the area of it with a triangle, but then we miss a small piece, etc. We will first work out this idea for the Titanic2.

The base of the grand triangle is 1. The base of the small triangle is $\frac{1}{4}$. The area of the small triangle is therefore smaller by a factor of 16. The even smaller triangle on top of that is a factor of 16 smaller, and so on. This gives a geometric sequence: $\sum_{k=0}^{n}\left(\frac{1}{16}\right)^k$. If the number of steps *n* goes to infinity, the sum becomes: $\frac{1}{1-\frac{1}{16}} = \frac{1}{\frac{15}{16}} = \frac{16}{15}$. This means that the area of the large triangle is multiplied by a factor $\frac{16}{15}$ to get the final enclosed area of the fractal above the *x*-axis. Multiplication by 2 gives the final enclosed area of the entire fractal.


y
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
x
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


y
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
x
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Titanic number

Now more generally: we introduce the *t* as Titanic number. With Titanic2, of course, $t = 2$, with Titanic3 $t = 3$, etc.
The number of lines in step 1 of the fractal is then $4t - 1$ and the ratio between the lengths of the line segments is $\sqrt{2t}$.

The factor by which the area of the triangle must be multiplied is: $\dfrac{(2t)^2}{(2t)^2 - 1}$.
The area of the triangle is: $\frac{1}{2} b \cdot h$. The base is 1 and for the triangle under the *x*-axis we multiply by 2. This means that the area of the two triangles (parallelogram) is equal to the height of the triangle and therefore equal to the *y*-coordinate of the highest point in step 1.

- In the case of Titanic2, the calculated *x*-coordinate of point B was: $\frac{1}{16}\sqrt{15}$.
  For the highest point, this value must be multiplied by $\frac{3}{2}$.
- In the case of Titanic3, the calculated *x*-coordinate of point B was: $\frac{1}{24}\sqrt{23}$.
  For the highest point, this value must be multiplied by $\frac{5}{3}$.
- In the case of Titanic4, the calculated *x*-coordinate of point B was: $\frac{1}{32}\sqrt{31}$.
  For the highest point, this value must be multiplied by $\frac{7}{4}$.

From this, the *y*-coordinate for all Titanic fractals can be derived: $\frac{1}{8t}\sqrt{8t - 1} \cdot \dfrac{2t - 1}{t}$.
This is also the formula for the area of the two triangles together.

All told, the formula for the final enclosed area of each Titanic fractal becomes:

$$O_t = \frac{\sqrt{8t-1}}{8t} \cdot \frac{2t-1}{t} \cdot \frac{(2t)^2}{(2t)^2-1} = \frac{\sqrt{8t-1}}{8t} \cdot \frac{2t-1}{t} \cdot \frac{4t^2}{(2t+1)(2t-1)} = \frac{\sqrt{8t-1}}{2(2t+1)}$$

For the above three Titanic fractals, this yields the following values:

$$O_2 = \tfrac{1}{10}\sqrt{15} \approx 0,38730$$

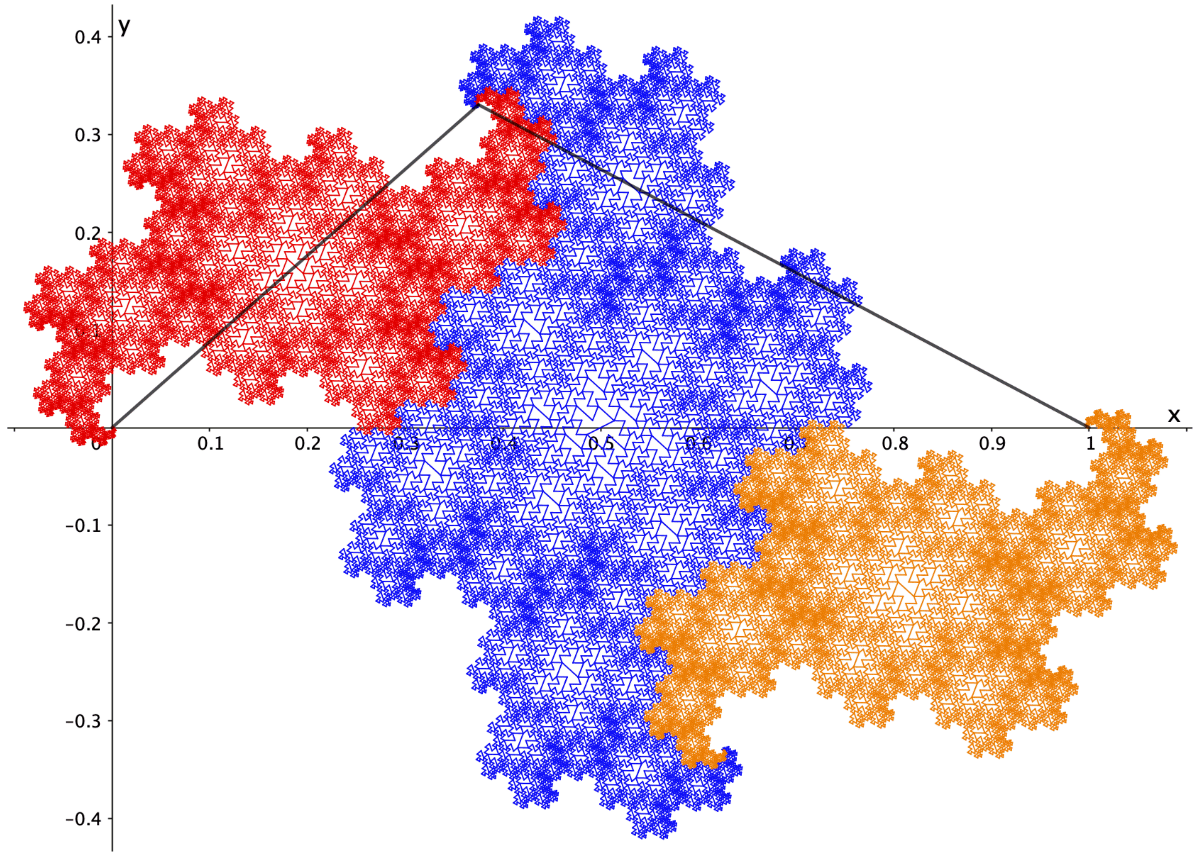
$$O_3 = \tfrac{1}{14}\sqrt{23} \approx 0,34256$$

$$O_4 = \tfrac{1}{18}\sqrt{31} \approx 0,30932$$

Because the Duck curve is constructed in exactly the same way, the same applies to the Duck curve: $t = 1$. The above surface formula also appears to be useful for the Duck curve:

$$O_1 = \tfrac{1}{6}\sqrt{7} \approx 0,44096$$

To be sure, let's take a look at the Duck curve with the triangle drawn by the highest point of step 1. On the right side, we see that the areas outside the triangle fit exactly into the white areas within the triangle. On the left side, exactly half of a scaled-down copy of the entire fractal protrudes outside the triangle. This can be added to the area of the triangle in the same way as with the Titanic fractals described above.



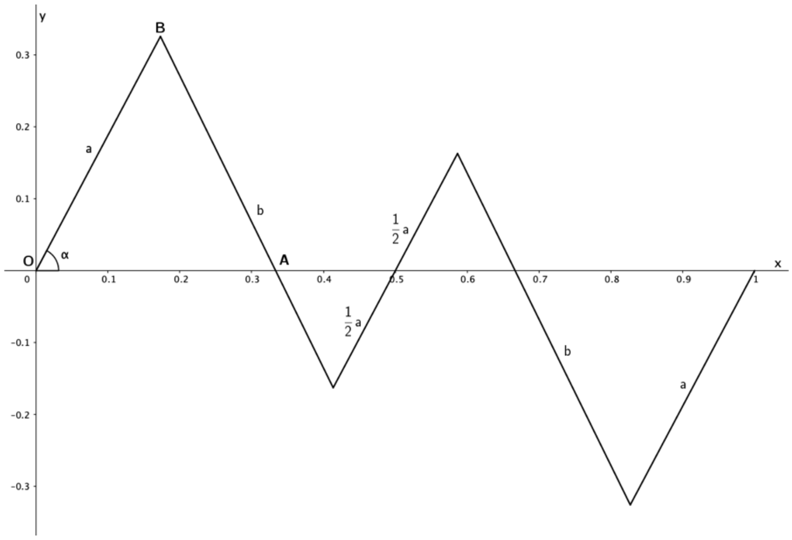
This confirms that the area formula also applies to the Duck curve.

Peaks

The Duck curve can be extended in another way: instead of three line segments, we take five, seven or nine line segments, as shown below. The line segments *a* are parallel and of equal length; the line segments *b* are also parallel and of equal length.

Twin Peaks

We start with five line segments. Because of symmetry, the middle line segment *a* is attached to the point in the middle $\left(\frac{1}{2}, 0\right)$.



For the dimension to be 2, $3a^2 + 2b^2 = 1$.
We again choose the ratio *k* between *a* and *b*: $a = kb$.
With this we express *a* and *b* in *k*:

$$3(kb)^2 + 2b^2 = 1 \quad \rightarrow \quad 3k^2b^2 + 2b^2 = 1 \quad \rightarrow \quad b^2(3k^2+2) = 1$$

$$b^2 = \frac{1}{3k^2+2} \quad \rightarrow \quad b = \frac{1}{\sqrt{3k^2+2}}$$

$$a = kb = \frac{k}{\sqrt{3k^2+2}}$$

The cosine rule in $\triangle OAB$ gives:

$$(\tfrac{2}{3}b)^2 = (\tfrac{1}{3})^2 + a^2 - 2 \cdot \tfrac{1}{3} \cdot a \cdot \cos\alpha$$

$$\frac{4}{9(3k^2+2)} = \frac{1}{9} + \frac{k^2}{3k^2+2} - \frac{2k}{3\sqrt{3k^2+2}}\cos\alpha$$

$$\frac{2k}{3\sqrt{3k^2+2}}\cos\alpha = \frac{1}{9} + \frac{k^2}{3k^2+2} - \frac{4}{9(3k^2+2)}$$

$$\cos\alpha = \left[\frac{3k^2+2}{9(3k^2+2)} + \frac{9k^2}{9(3k^2+2)} - \frac{4}{9(3k^2+2)}\right]\frac{3\sqrt{3k^2+2}}{2k}$$

$$= \frac{12k^2-2}{9(3k^2+2)} \cdot \frac{3\sqrt{3k^2+2}}{2k} = \frac{6k^2-1}{3k\sqrt{3k^2+2}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{6k^2-1}{3k\sqrt{3k^2+2}}\right)^2} = \sqrt{1-\frac{(6k^2-1)^2}{9k^2(3k^2+2)}}$$

$$= \sqrt{\frac{9k^2(3k^2+2)}{9k^2(3k^2+2)} - \frac{36k^4-12k^2+1}{9k^2(3k^2+2)}} = \sqrt{\frac{27k^4+18k^2-36k^4+12k^2-1}{9k^2(3k^2+2)}}$$

$$= \frac{\sqrt{-9k^4+30k^2-1}}{3k\sqrt{3k^2+2}}$$

$$x_B = a\cdot\cos\alpha = \frac{k}{\sqrt{3k^2+2}}\cdot\frac{6k^2-1}{3k\sqrt{3k^2+2}} = \frac{6k^2-1}{3(3k^2+2)}$$

$$y_B = a\cdot\sin\alpha = \frac{k}{\sqrt{3k^2+2}}\cdot\frac{\sqrt{-9k^4+30k^2-1}}{3k\sqrt{3k^2+2}} = \frac{\sqrt{-9k^4+30k^2-1}}{3(3k^2+2)}$$

$$k = \sqrt{\frac{1}{3}}$$

For $k=\sqrt{\frac{1}{3}}$ we get:

$$a = \frac{\frac{1}{\sqrt{3}}}{\sqrt{3\cdot\left(\frac{1}{\sqrt{3}}\right)^2+2}} = \frac{1}{3}$$

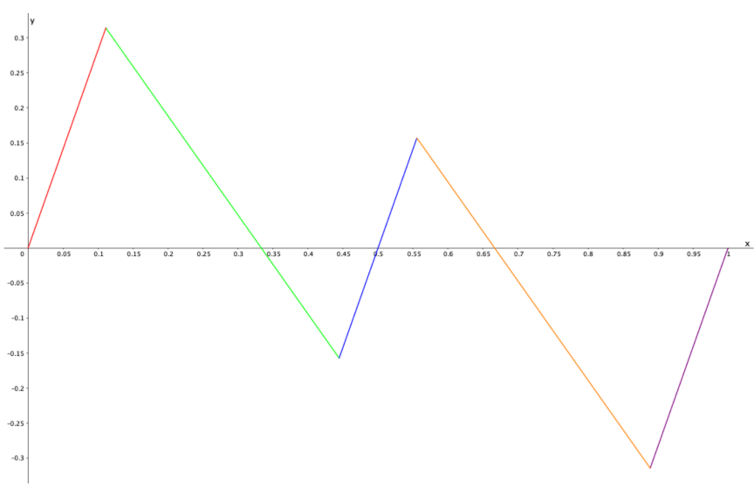
$$b = \frac{1}{\sqrt{3\cdot\left(\frac{1}{\sqrt{3}}\right)^2+2}} = \frac{1}{\sqrt{3}}$$

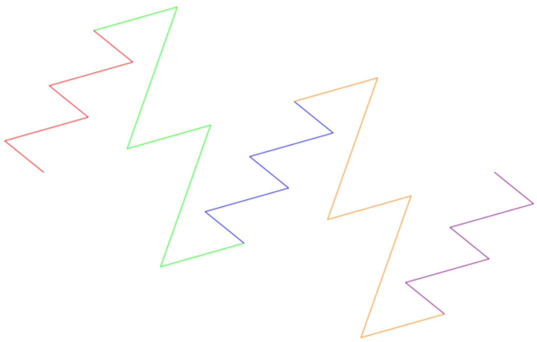
$$x_B = \frac{6\left(\frac{1}{\sqrt{3}}\right)^2-1}{3\left(3\left(\frac{1}{\sqrt{3}}\right)^2+2\right)} = \frac{2-1}{3(1+2)} = \frac{1}{9}$$

$$y_B = \frac{\sqrt{-9\left(\frac{1}{\sqrt{3}}\right)^4+30\left(\frac{1}{\sqrt{3}}\right)^2-1}}{3\left(3\left(\frac{1}{\sqrt{3}}\right)^2+2\right)} = \frac{\sqrt{-1+10-1}}{3(1+2)} = \frac{\sqrt{8}}{9} = \frac{2}{9}\sqrt{2}$$

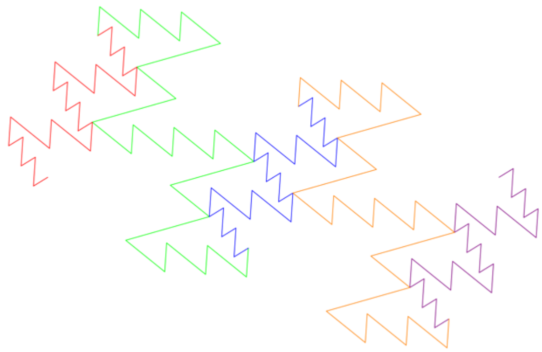
The coordinates of the vertices are resp.: $(0, 0)$, $\left(\frac{1}{9}, \frac{2}{9}\sqrt{2}\right)$, $\left(\frac{4}{9}, -\frac{1}{9}\sqrt{2}\right)$, $\left(\frac{5}{9}, \frac{1}{9}\sqrt{2}\right)$, $\left(\frac{8}{9}, -\frac{2}{9}\sqrt{2}\right)$ and $(1, 0)$.

Step 1 looks like the one on the right.



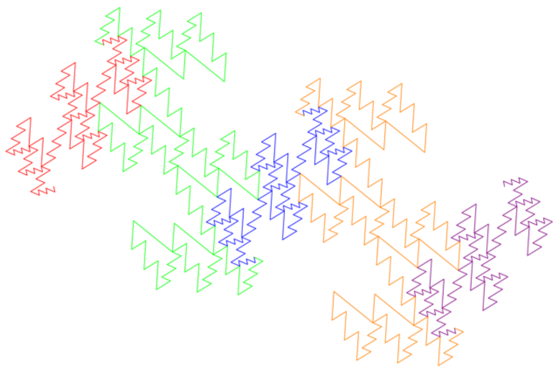


step 2

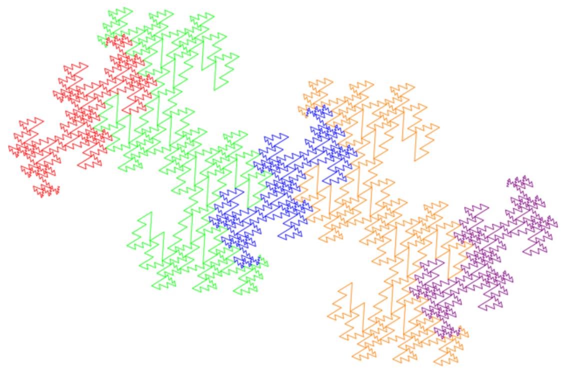


step 3

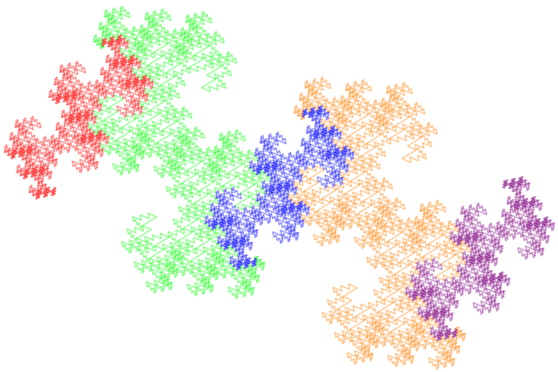
In step 3 we see that the fractal touches itself. In the enclosed form at the top, you can see a mountain landscape with some imagination, with two mountain peaks on the right and a conifer tree on the left. That's why I call this fractal **Twin Peaks** (after the 1990 TV series of the same name).
The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



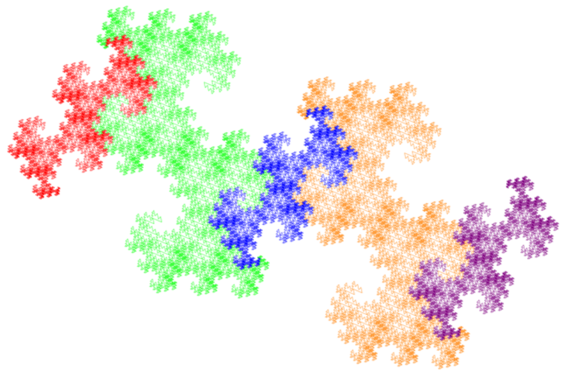
step 4



step 5



step 6

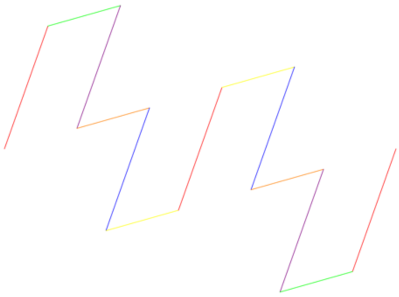


step 7

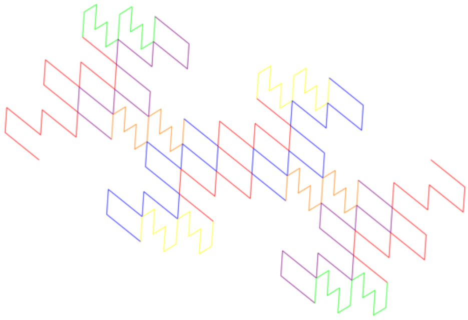
Twin Peaks variant

As with the Duck curve, we can see if we can make a variant of the Twin Peaks fractal, which is also plane-filling. We replace the longest two line segments from step 1 with the same step 1.

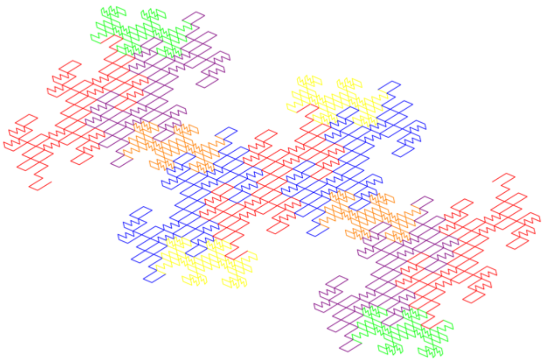
In step 4 it can be seen that the contours of this fractal become the same as the contours of the Twin Peaks fractal. The final enclosed surfaces will be equal to each other.



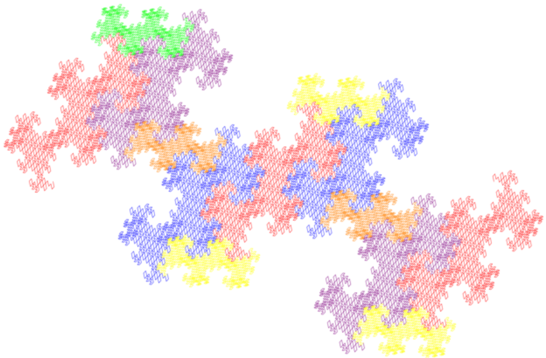
step 1



step 2



step 3



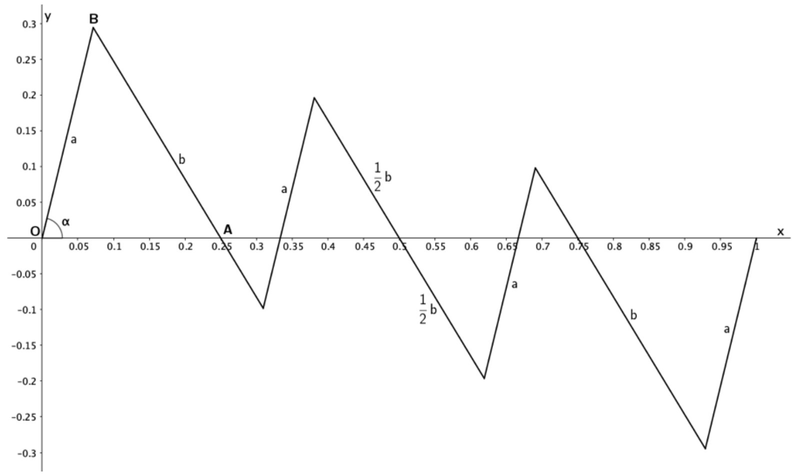
step 4

There are probably many more variants of the Twin Peaks fractal to be found, in the same way as with the Duck curve. The number of line segments in step 1 of the variants increases rapidly, because each replacement of a line segment gives four extra line segments.
I will limit myself here to this one example.

Triplet Peaks

We start with seven line segments. Because of symmetry, the middle line segment *b* is attached to the point in the middle $\left(\frac{1}{2}, 0\right)$.

For the dimension to be 2, $4a^2 + 3b^2 = 1$.
We again choose the ratio *k* between *a* and *b*: $a = kb$.
With this we express *a* and *b* in *k*:



$4(kb)^2+3b^2=1 \quad \rightarrow \quad 4k^2b^2+3b^2=1 \quad \rightarrow \quad b^2(4k^2+3)=1$

$$b^2=\frac{1}{4k^2+3} \quad \rightarrow \quad b=\frac{1}{\sqrt{4k^2+3}}$$

$$a=kb=\frac{k}{\sqrt{4k^2+3}}$$

The cosine rule in $\triangle OAB$ gives:

$$(\tfrac{3}{4}b)^2=(\tfrac{1}{4})^2+a^2-2\cdot\tfrac{1}{4}\cdot a\cdot\cos\alpha$$

$$\frac{9}{16\left(4k^2+3\right)}=\tfrac{1}{16}+\frac{k^2}{4k^2+3}-\frac{k}{2\sqrt{4k^2+3}}\cos\alpha$$

$$\frac{k}{2\sqrt{4k^2+3}}\cos\alpha=\tfrac{1}{16}+\frac{k^2}{4k^2+3}-\frac{9}{16\left(4k^2+3\right)}$$

$$\cos\alpha=\left[\frac{4k^2+3}{16(4k^2+3)}+\frac{16k^2}{16(4k^2+3)}-\frac{9}{16\left(4k^2+3\right)}\right]\frac{2\sqrt{4k^2+3}}{k}$$

$$=\frac{20k^2-6}{16(4k^2+3)}\cdot\frac{2\sqrt{4k^2+3}}{k}=\frac{10k^2-3}{4k\sqrt{4k^2+3}}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{10k^2-3}{4k\sqrt{4k^2+3}}\right)^2}=\sqrt{1-\frac{(10k^2-3)^2}{16k^2(4k^2+3)}}$$

$$=\sqrt{\frac{64k^4+48k^2}{16k^2(4k^2+3)}-\frac{100k^4-60k^2+9}{16k^2(4k^2+3)}}=\frac{1}{4k}\sqrt{\frac{64k^4+48k^2-100k^4+60k^2-9}{4k^2+3}}$$

$$=\frac{\sqrt{-36k^4+108k^2-9}}{4k\sqrt{4k^2+3}}$$

$$x_B=a\cdot\cos\alpha=\frac{k}{\sqrt{4k^2+3}}\cdot\frac{10k^2-3}{4k\sqrt{4k^2+3}}=\frac{10k^2-3}{4(4k^2+3)}$$

$$y_B=a\cdot\sin\alpha=\frac{k}{\sqrt{4k^2+3}}\cdot\frac{\sqrt{-36k^4+108k^2-9}}{4k\sqrt{4k^2+3}}=\frac{\sqrt{-36k^4+108k^2-9}}{4(4k^2+3)}$$

$$k=\sqrt{\tfrac{1}{4}}$$

For $k=\frac{1}{2}$ we get:

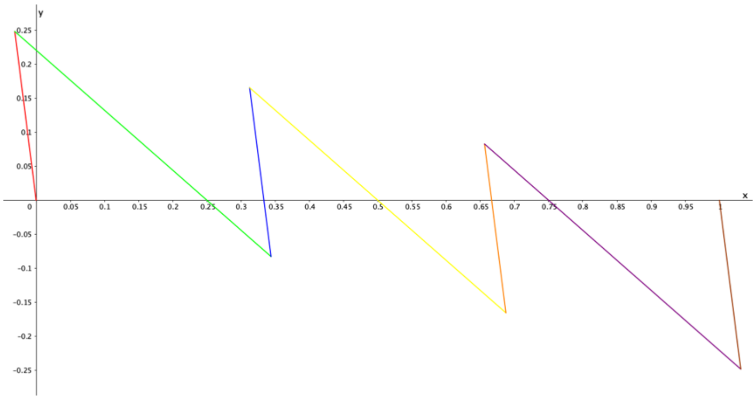
$$a=\frac{\frac{1}{2}}{\sqrt{4\left(\frac{1}{2}\right)^2+3}}=\tfrac{1}{4}$$

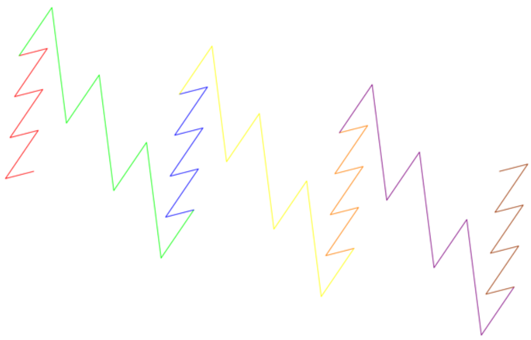
$$b=\frac{1}{\sqrt{4\left(\frac{1}{2}\right)^2+3}}=\tfrac{1}{2}$$

$$x_B = \frac{10\left(\frac{1}{2}\right)^2 - 3}{4\left(4\left(\frac{1}{2}\right)^2 + 3\right)} = \frac{-\frac{1}{2}}{16} = -\frac{1}{32}$$

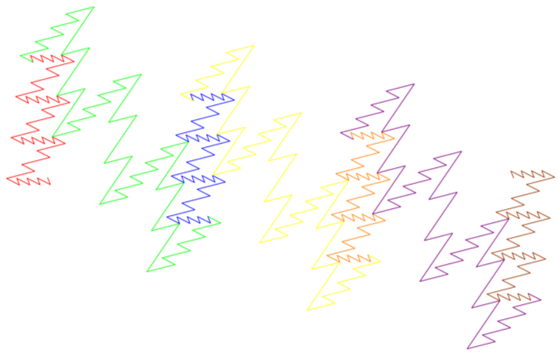
$$y_B = \frac{\sqrt{-36\left(\frac{1}{2}\right)^4 + 108\left(\frac{1}{2}\right)^2 - 9}}{4\left(4\left(\frac{1}{2}\right)^2 + 3\right)} = \frac{\sqrt{-\frac{9}{4} + 27 - 9}}{16} = \frac{\sqrt{\frac{63}{4}}}{16} = \frac{3}{32}\sqrt{7}$$

The coordinates of the vertices are resp.: $(0, 0)$, $\left(-\frac{1}{32}, \frac{3}{32}\sqrt{7}\right)$, $\left(\frac{11}{32}, -\frac{1}{32}\sqrt{7}\right)$, $\left(\frac{10}{32}, \frac{2}{32}\sqrt{7}\right)$, $\left(\frac{22}{32}, -\frac{2}{32}\sqrt{7}\right)$, $\left(\frac{21}{32}, \frac{1}{32}\sqrt{7}\right)$, $\left(\frac{33}{32}, -\frac{3}{32}\sqrt{7}\right)$ and $(1, 0)$.
Step 1 looks like the one on the right.



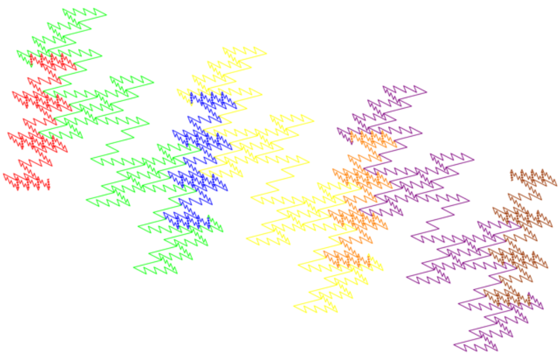


step 2

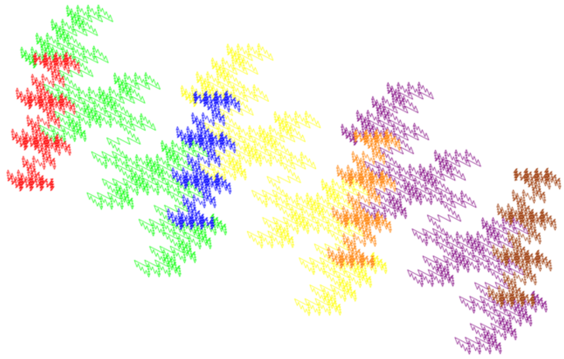


step 3

In step 3 we see that the fractal touches itself. In the enclosed form at the top, you can see a mountain landscape with some imagination, with three mountain peaks on the right and a coniferous tree on the left. That's why I call these fractal **Triplet Peaks**.
The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.



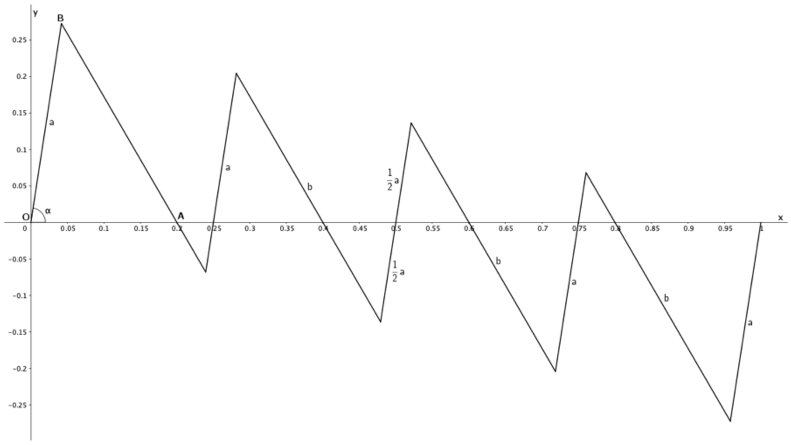
step 4



step 5

Quads Peaks

We start with nine line segments. Because of symmetry, the middle line segment *a is* attached to the point in the middle $\left(\frac{1}{2}, 0\right)$.



For the dimension to be 2, $5a^2 + 4b^2 = 1$.

We again choose the ratio *k* between *a* and *b*: $a = kb$. With this we express *a* and *b* in *k*:

$$5(kb)^2 + 4b^2 = 1 \quad \rightarrow \quad 5k^2b^2 + 4b^2 = 1 \quad \rightarrow \quad b^2(5k^2+4) = 1$$

$$b^2 = \frac{1}{5k^2+4} \quad \rightarrow \quad b = \frac{1}{\sqrt{5k^2+4}}$$

$$a = kb = \frac{k}{\sqrt{5k^2+4}}$$

The cosine rule in $\triangle OAB$ gives:

$$\left(\tfrac{4}{5}b\right)^2 = \left(\tfrac{1}{5}\right)^2 + a^2 - 2\cdot\tfrac{1}{5}\cdot a\cdot\cos\alpha$$

$$\frac{16}{25\left(5k^2+4\right)} = \frac{1}{25} + \frac{k^2}{5k^2+4} - \frac{2k}{5\sqrt{5k^2+4}}\cos\alpha$$

$$\frac{2k}{5\sqrt{5k^2+4}}\cos\alpha = \frac{1}{25} + \frac{k^2}{5k^2+4} - \frac{16}{25\left(5k^2+4\right)}$$

$$\cos\alpha = \left[\frac{5k^2+4}{25(5k^2+4)} + \frac{25k^2}{25(5k^2+4)} - \frac{16}{25\left(5k^2+4\right)}\right]\frac{5\sqrt{5k^2+4}}{2k}$$

$$= \frac{30k^2-12}{25\left(5k^2+4\right)}\cdot\frac{5\sqrt{5k^2+4}}{2k} = \frac{15k^2-6}{5k\sqrt{5k^2+4}}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{15k^2-6}{5k\sqrt{5k^2+4}}\right)^2} = \sqrt{1-\frac{(15k^2-6)^2}{25k^2(5k^2+4)}}$$

$$= \sqrt{\frac{125k^4+100k^2}{25k^2(5k^2+4)} - \frac{225k^4-180k^2+36}{25k^2(5k^2+4)}} = \frac{1}{5k}\sqrt{\frac{125k^4+100k^2-225k^4+180k^2-36}{5k^2+4}}$$

$$= \frac{\sqrt{-100k^4+280k^2-36}}{5k\sqrt{5k^2+4}}$$

$$x_B = a\cdot\cos\alpha = \frac{k}{\sqrt{5k^2+4}}\cdot\frac{15k^2-6}{5k\sqrt{5k^2+4}} = \frac{15k^2-6}{5(5k^2+4)}$$

$$y_B = a\cdot\sin\alpha = \frac{k}{\sqrt{5k^2+4}}\cdot\frac{\sqrt{-100k^4+280k^2-36}}{5k\sqrt{5k^2+4}} = \frac{\sqrt{-100k^4+280k^2-36}}{5(5k^2+4)}$$

$$k = \sqrt{\frac{1}{5}}$$

For $k = 1/\sqrt{5}$ we get:

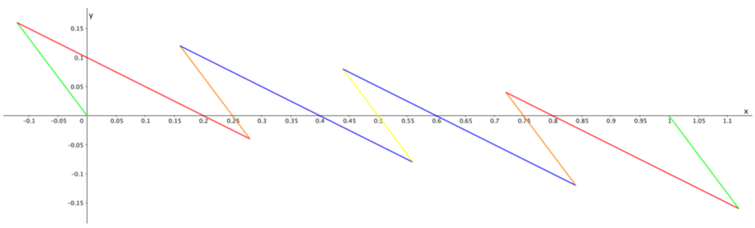
$$a = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5\left(\frac{1}{\sqrt{5}}\right)^2+4}} = \frac{1}{5}$$

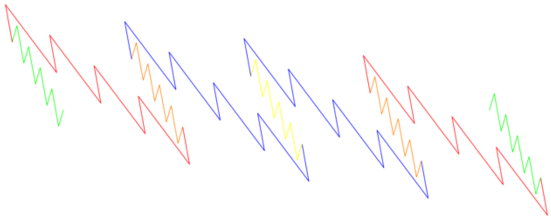
$$b = \frac{1}{\sqrt{5\left(\frac{1}{\sqrt{5}}\right)^2+4}} = \frac{1}{\sqrt{5}}$$

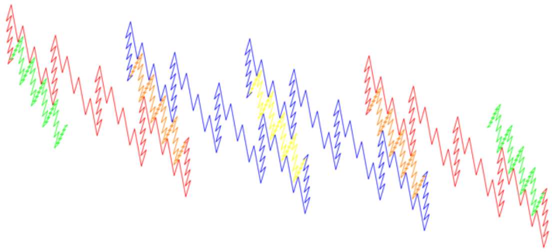
$$x_B = \frac{15\left(\frac{1}{\sqrt{5}}\right)^2-6}{5\left(5\left(\frac{1}{\sqrt{5}}\right)^2+4\right)} = \frac{-3}{25} = -\frac{3}{25}$$

$$y_B = \frac{\sqrt{-100\left(\frac{1}{\sqrt{5}}\right)^4+280\left(\frac{1}{\sqrt{5}}\right)^2-36}}{5\left(5\left(\frac{1}{\sqrt{5}}\right)^2+4\right)} = \frac{\sqrt{-4+56-36}}{25} = \frac{\sqrt{16}}{25} = \frac{4}{25}$$

The coordinates of the vertices are resp.: $(0, 0)$, $\left(-\frac{3}{25}, \frac{4}{25}\right)$, $\left(\frac{7}{25}, -\frac{1}{25}\right)$, $\left(\frac{4}{25}, \frac{3}{25}\right)$, $\left(\frac{14}{25}, -\frac{2}{25}\right)$, $\left(\frac{11}{25}, \frac{2}{25}\right)$, $\left(\frac{21}{25}, -\frac{3}{25}\right)$, $\left(\frac{18}{25}, \frac{1}{25}\right)$, $\left(\frac{28}{25}, -\frac{4}{25}\right)$ and $(1, 0)$.
Step 1 looks like the one on the right.



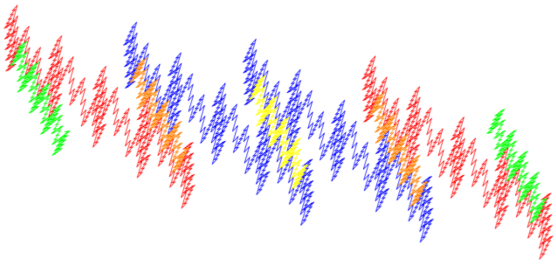


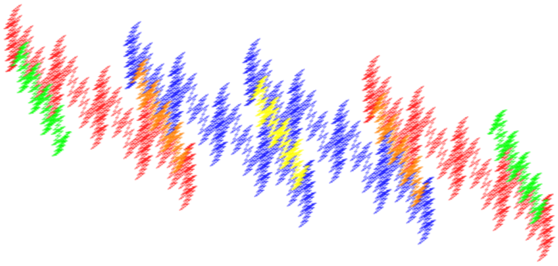


step 2

step 3

In step 3 we see that the fractal touches itself. In the enclosed form at the top, you can see a mountain landscape with some imagination, with four mountain peaks on the right and a coniferous tree on the left (unfortunately hardly visible anymore...). That's why I call these fractal **Quads Peaks**.
The dimension of this fractal is 2 and there are no intersecting or overlapping line segments. The fractal is therefore plane-filling.





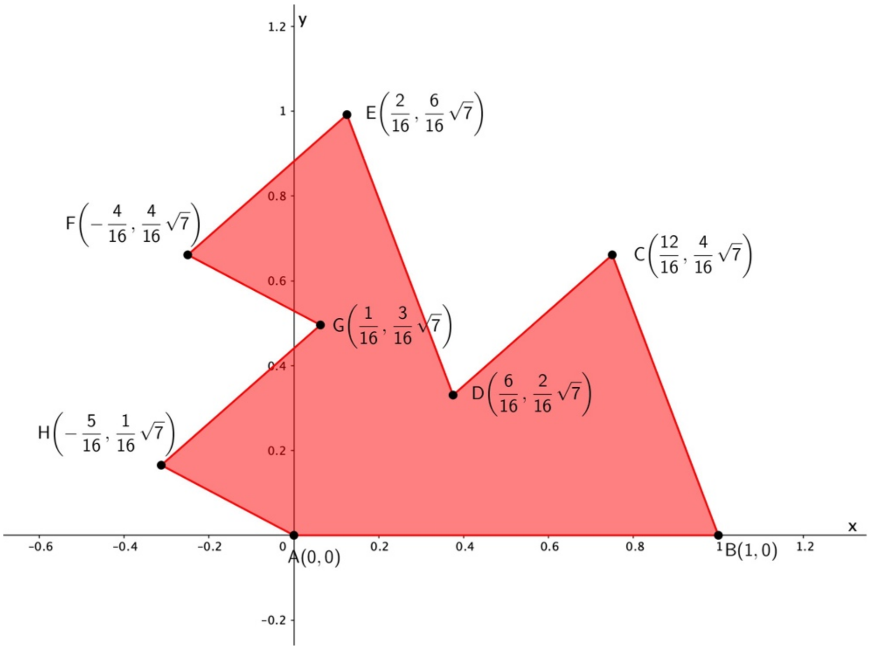
step 4

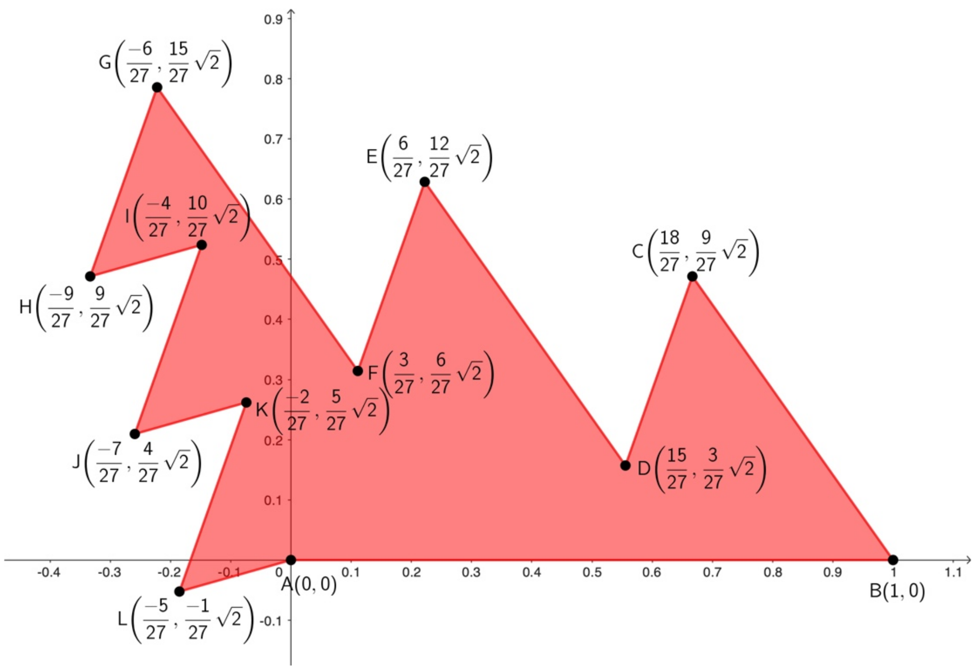
step 5

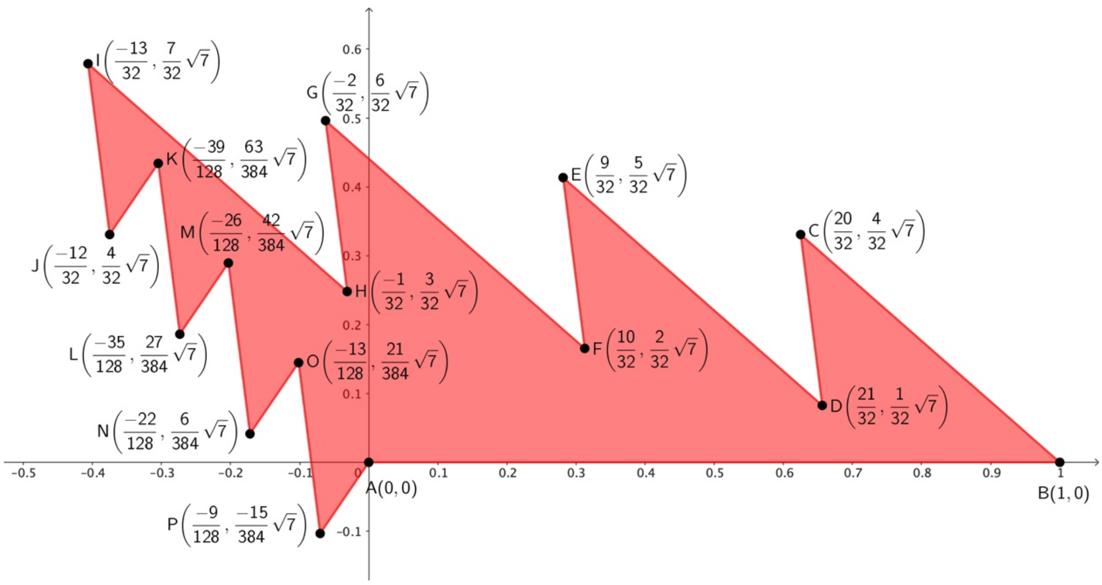
This fractal doesn't get any prettier. It is the last in this series, because a comparable fractal with five mountain peaks can no longer be constructed.

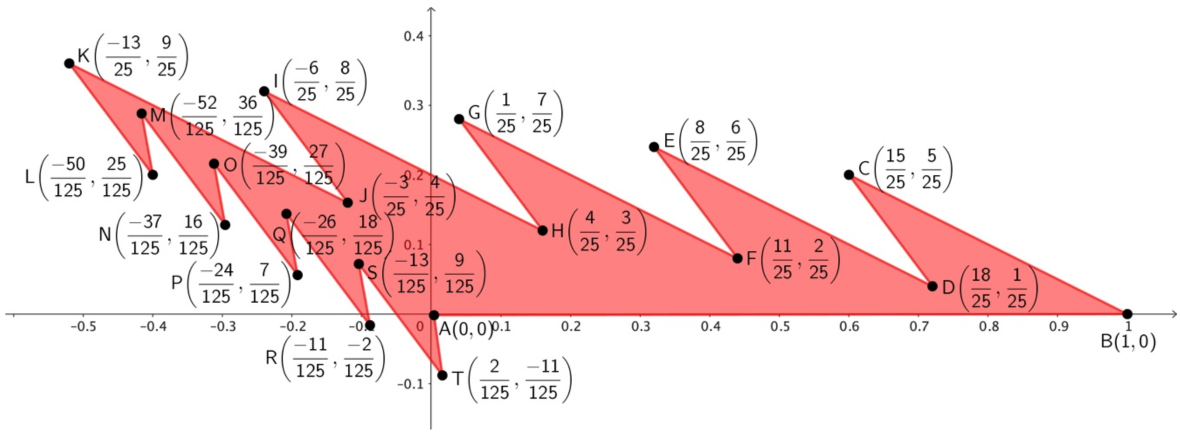
Peaks figuren

The enclosed figures of the Duck Curve, Twin Peaks, Triplet Peaks and Quads Peaks look like this, with the coordinates of the vertices:

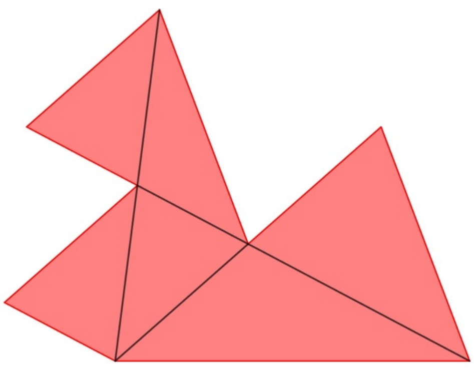

y
x
$E\left(\frac{2}{16}, \frac{6}{16}\sqrt{7}\right)$
$F\left(-\frac{4}{16}, \frac{4}{16}\sqrt{7}\right)$
$C\left(\frac{12}{16}, \frac{4}{16}\sqrt{7}\right)$
$G\left(\frac{1}{16}, \frac{3}{16}\sqrt{7}\right)$
$D\left(\frac{6}{16}, \frac{2}{16}\sqrt{7}\right)$
$H\left(-\frac{5}{16}, \frac{1}{16}\sqrt{7}\right)$
A(0, 0)
B(1, 0)

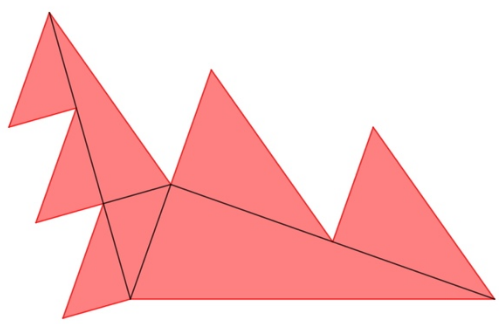

$G\left(\frac{-6}{27}, \frac{15}{27}\sqrt{2}\right)$
$E\left(\frac{6}{27}, \frac{12}{27}\sqrt{2}\right)$
$I\left(\frac{-4}{27}, \frac{10}{27}\sqrt{2}\right)$
$C\left(\frac{18}{27}, \frac{9}{27}\sqrt{2}\right)$
$H\left(\frac{-9}{27}, \frac{9}{27}\sqrt{2}\right)$
$F\left(\frac{3}{27}, \frac{6}{27}\sqrt{2}\right)$
$K\left(\frac{-2}{27}, \frac{5}{27}\sqrt{2}\right)$
$J\left(\frac{-7}{27}, \frac{4}{27}\sqrt{2}\right)$
$D\left(\frac{15}{27}, \frac{3}{27}\sqrt{2}\right)$
A(0, 0)
B(1, 0)
$L\left(\frac{-5}{27}, \frac{-1}{27}\sqrt{2}\right)$

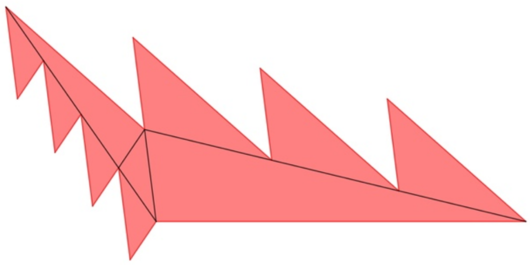


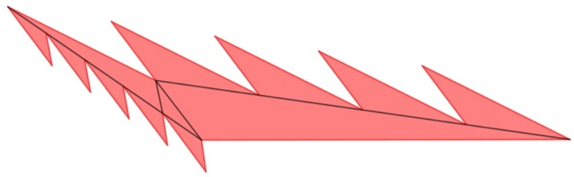


All four of these figures can be divided into triangles in a similar way:

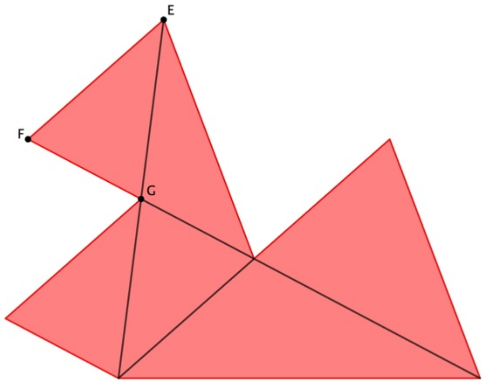


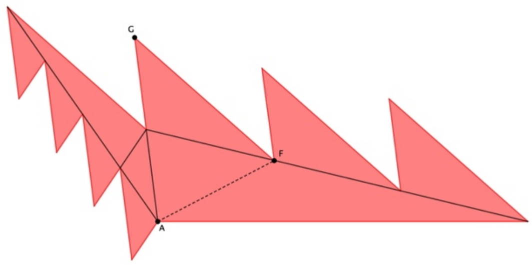






Now two things stand out:





1. The triangle FGE of the Duck figure and triangle AFG of the Triplet Peaks are exactly the same size and have the same shape. They are isosceles triangles with sides with length $\frac{1}{2}$ and $\frac{1}{4}\sqrt{2}$. In the Duck figure, the outer angles are equal to the base angle of this isosceles triangle and in the Triplet Peaks, the outer angles are equal to the apex angle.

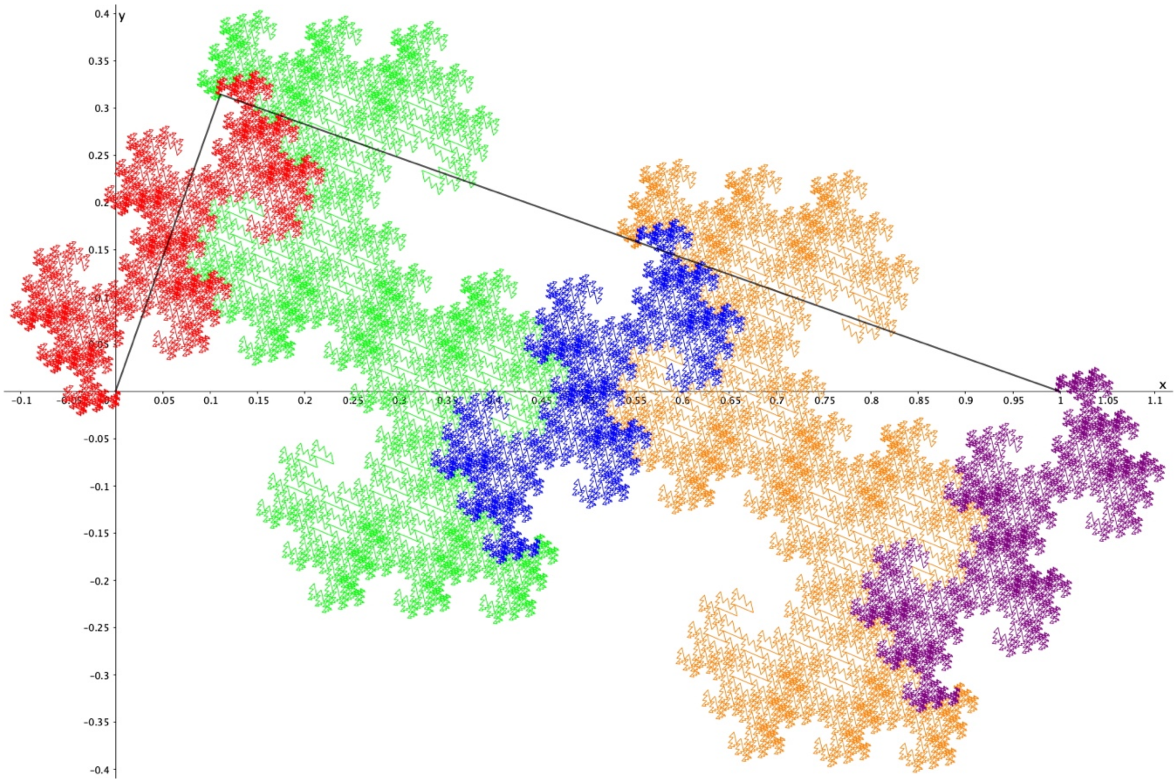
2. At the Twin Peaks, AD and ED are both $\frac{1}{3}\sqrt{3}$ long. AF and FE are both $\frac{1}{3}$ long. That means that $\angle$ AFD and $\angle$ EFD both are 90º. FD is $\frac{1}{3}\sqrt{2}$ long.
   This figure therefore consists of all right triangles, in the proportion 1, $\sqrt{2}$ and $\sqrt{3}$.

Enclosed area

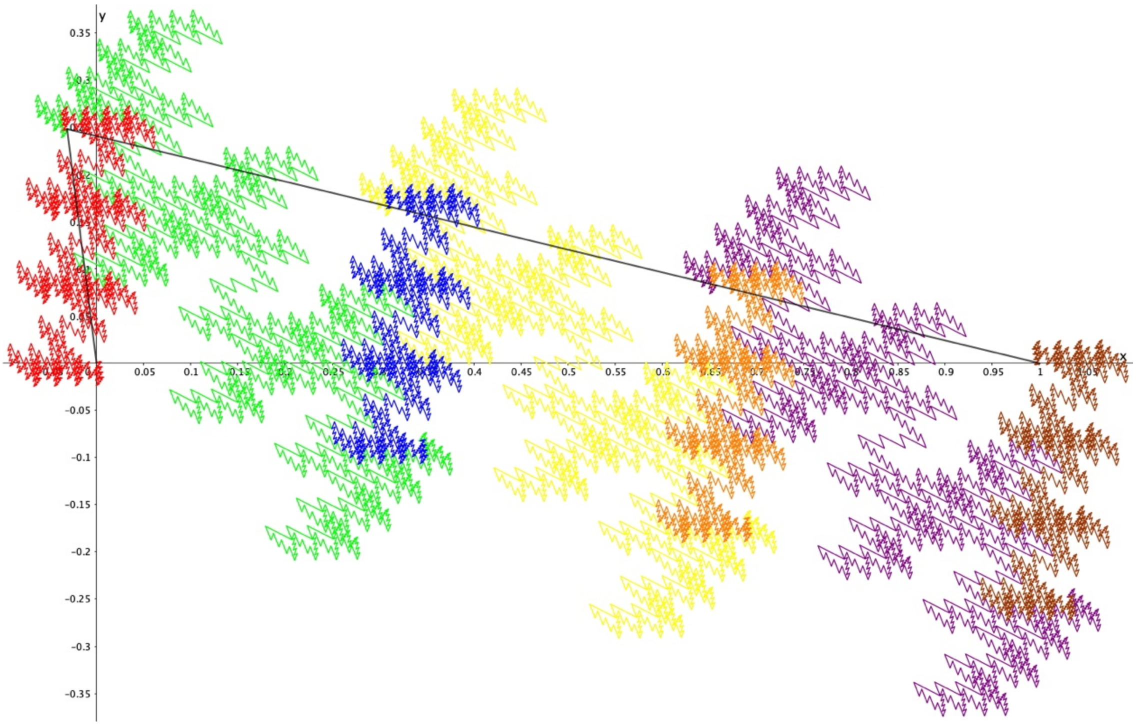
As with the Titanic fractals, I want to look at the enclosed surface of the Peaks fractals. I follow the same method.

From the Twin Peaks we take step 7 and draw two lines from the highest point of step 1 to $(0, 0)$ and to $(1, 0)$. Together with the *x*-axis, this creates a triangle that roughly corresponds to the shape of the fractal above the *x*-axis.
On the right side we see two pieces of the fractal falling outside the triangle, and also two white pieces inside the triangle. If we were to let the steps go to infinity, the areas

of the areas inside and outside the triangle would be exactly the same size. For the area determination we can use the straight line of the triangle on the right.

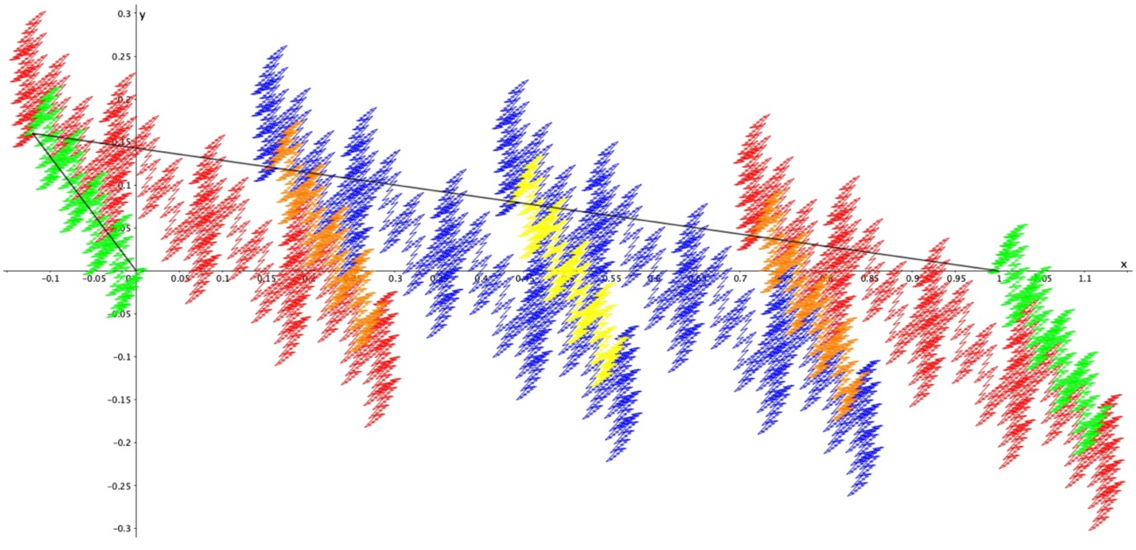


On the left side we see that half of a reduced fractal remains outside the triangle. Here we have to make another correction.
The height of the triangle is $\frac{2}{9}\sqrt{2}$. The area of the triangle is therefore $\frac{1}{2} \cdot \frac{2}{9}\sqrt{2} \cdot 1 = \frac{1}{9}\sqrt{2}$. The length of the left line segment of the triangle is $\frac{1}{3}$, so the area of the reduced fractal is a factor $\frac{1}{9}$ smaller. The correction is: $\frac{1}{1-\frac{1}{9}} = \frac{1}{\frac{8}{9}} = \frac{9}{8}$.
After another multiplication by 2 for the part below the *x*-axis, the area of the Twin Peaks becomes after an infinite number of steps: $2 \cdot \frac{1}{9}\sqrt{2} \cdot \frac{9}{8} = \frac{1}{4}\sqrt{2} \approx 0,35355$ ·



For the Triplet Peaks, the calculation of the area after an infinite number of steps:

$$2 \cdot \frac{3}{64}\sqrt{7} \cdot \frac{16}{15} = \frac{1}{10}\sqrt{7} \approx 0,26458 \cdot$$



For the Quads Peaks, the calculation of the area after an infinite number of steps:

$$2 \cdot \frac{4}{50} \cdot \frac{25}{24} = \frac{1}{6} \approx 0,16667 \cdot$$

Conclusion

These were my research questions:

1. Are there also plane-filling fractals that are not tied to a triangular or square grid?
2. If so, what are the properties of such fractals?

I can answer the first question with a resounding "yes". Based on Mandelbrot's suggestion on how to calculate the dimension of fractals that have line segments of unequal length in step 1, I reversed this method and fixed the dimension at 2. That is one of the conditions for a plane-filling fractal: its dimension is 2. The other condition is symmetry. In this way I found the Duck curve and it also turned out to contain an infinite number of variants.
In addition, two other series of other fractals have been found in a similar way: the Titanic fractals and the Peaks fractals.

Properties of the Duck curve and its variants are:

- The ratio between the lengths of the line segments is $1{:}\sqrt{2}$.
- For all variants and for all steps, line segments of equal length are also parallel to each other.
- In addition to the Duck figure, the bird and its nest (parallelogram), more and more different enclosed forms are emerging.
- The final contours and enclosed area of all variants are the same as those of the Duck curve.
- The lengths of the line segments in step 1 differ by one or more factors $\sqrt{2}$ from each other. The corresponding colored subsurfaces in the final fractal differ from each other by the same number of factors 2.

The Titanic fractals form a series of new fractals. Initially, I called them Titanic2, Titanic3 and Titanic4, with the number referring to the number of "funnels" on the sinking ship. Afterwards I defined this number as the Titanic number *t*, which can be used to describe a number of properties for all Titanic fractals. For the Duck curve, the following applies $t = 1$.
Features of the Titanic fractals series are:

- The number of line segments in step 1 of the fractal is $4t - 1$.
- The ratio between the lengths of the line segments is $1{:}\sqrt{2t}$.
- Over all steps within a fractal, line segments of equal length are also parallel to each other.
- More different enclosed forms are emerging.
- The enclosed surface of the fractal if this fractal were to be carried through to infinity is $O_t = \frac{\sqrt{8t-1}}{2(2t+1)}$.

- The lengths of the line segments in step 1 differ by a factor $\sqrt{2t}$ from each other. The corresponding colored subsurfaces in the final fractal differ by a factor $2t$ from each other.

The Peaks fractals form three new fractals. I have called them Twin Peaks, Triplet Peaks and Quads Peaks. You could say that the Duck curve also fits in this series, with a mountain top on the right and a coniferous tree on the left.
Characteristics of the Peaks fractals are:

- The enclosed figures are all uniform per fractal.
- The ratio between the lengths of the line segments is resp. $1{:}\sqrt{3}$, $1{:}\sqrt{4}$ and $1{:}\sqrt{5}$.
- Over all steps within a fractal, line segments of equal length are also parallel to each other.

Reflection

In this study, the discovery of the Duck curve and its variants was discussed. Only one variant of the Titanic2 fractal has been described; there are probably many more. There are probably also many variants of the Titanic3 and Titanic4 fractals.
In addition, Titanic fractals and their variants can be found with a $t > 4$.
Of the three Peaks fractals, only a variant has been described for the Twin Peaks. Perhaps there are many more variants to be found there as well.

It cannot be ruled out that there are more plane-filling fractals that are not tied to a triangular or square grid. It may be the subject of further research to find it. The methods described in this report can be helpful in this regard.

Epilogue

What started with an attempt to generate fractals with the computer program GeoGebra, has culminated in a study on fractals that I have not come across in previous research on the subject. It has regularly given me a kick, about working with GeoGebra, about the mathematical discoveries I made and about the beautiful visualizations I made.
Many of those visualizations are in this report; many others don't either. This concerns visualizations that do not relate to the subject of this report, but also animations that are related to this topic, but for which this paper is not the right medium.

Pictures say more than a thousand words. And that applies even more to moving pictures, or animations. Last year I participated in an art route in my residential area. In addition to works of art derived from mathematics – a copy of which is included in this report – I have shown fractals on canvas and also animations with fractals on a number of computer screens. The latter in particular were very much in the spotlight. Regularity and symmetry appeal to people, even without awareness of the mathematics behind it.

I started this project out of my curiosity about whether GeoGebra can be used to make images of fractals. I had already worked a lot with GeoGebra from my work as a mathematics teacher and I found it a pleasant program to work with. But with this project I ran into the limits of the program.
First of all, there was the limited length of the scripts that could be entered. That was difficult, but it also forced me to write more compact scripts. Worse was the limited length of lists of (vertice) points: a maximum of about 8000 points fit in such a list. If you make the list longer, GeoGebra does not give an error message, but it suddenly becomes very slow... That requires some tricks to work around that.
In hindsight, I might have been better off using a programming language to make images of fractals.

Bibliography

Mandelbrot, B., *The Fractal Geometry of Nature*, W.H. Freeman and Co., San Francisco, 1982.

Ventrella, Jeffrey J., *Brain-Filling Curves: A Fractal Bestiary*, LuLu.com, (2012). URL: http://archive.org/download/BrainfillingCurves-AFractalBestiary/.

Haverkort, Herman, Nico Bakker's duck plane-filling curve, (2022). URL: http://herman.haverkort.net/_media/research/eend.pdf